EE 209 Lecture Notes:

P1

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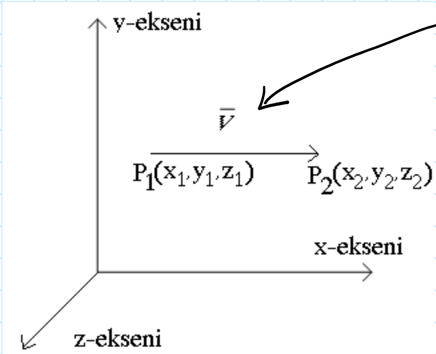
Vector Calculus:

Vectors:

A vector is a quantity with a magnitude and direction.

Electric and magnetic fields are vectors.

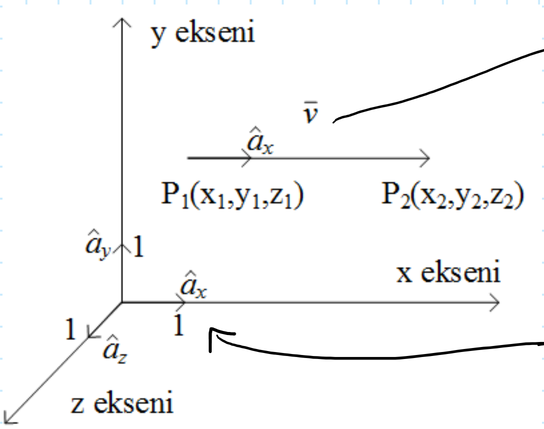
$\bar{V}$ is a vector.



Unit vector: It is a vector with unity magnitude

$\|\bar{a}\|$ = Norm of a vector = Magnitude of a vector

For unit vector $\|\bar{a}\| = |\bar{a}| = 1$.



$$\bar{V} = \hat{a}_x \|\bar{V}\|$$

direction magnitude.

$$\|\bar{V}\| = \left[(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2\right]^{\frac{1}{2}}$$

unit vectors in the x, y and z directions.

The vector $\bar{V}$ in the above figure is directed on the x-axis. Thus,

$$\bar{V} = \hat{a}_x \cdot \underbrace{V_x}_{\text{magnitude along the x-direction}}$$

Any vector in 3D space can be composed by components.

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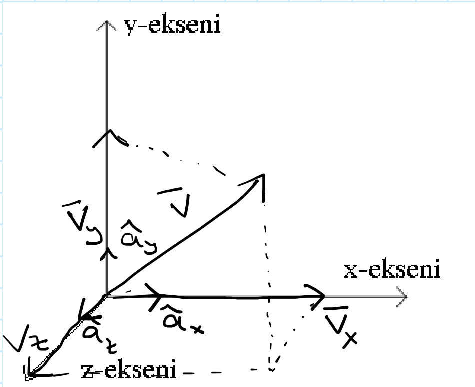


$$\bar{V} = \underbrace{\bar{V}_x + \bar{V}_y + \bar{V}_z}_{\text{components.}}$$

where

$$\bar{V}_x = \hat{a}_x V_x \,, \quad \bar{V}_y = \hat{a}_y V_y \,, \quad \bar{V}_z = \hat{a}_z V_z$$

Thus,

$$\bar{V} = \hat{a}_x V_x + \hat{a}_y V_y + \hat{a}_z V_z \quad \checkmark$$

$$\bar{V} = \langle V_x , V_y , V_z \rangle$$ (notation used by mathematicians)

Differentials (scalar, vector):

In a 3D space, to show small distances, we use the following notations:

$$\Delta x = x_2 - x_1 = \text{Small distance in } x \text{ direction}$$

$$\Delta x$$

$$0 \quad x_1 \quad x_2 \quad \longrightarrow x$$

Similarly,

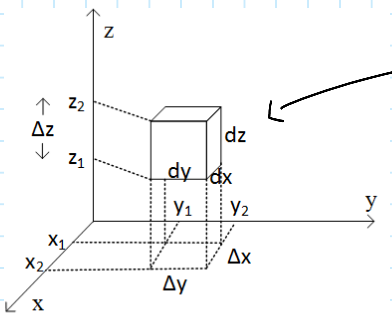
$$\Delta y = y_2 - y_1 \,, \quad \Delta z = z_2 - z_1 \,.$$

Define differentials as:

$$dx = \lim_{(x_2 - x_1) \to 0} \Delta x \,,$$

$$dy = \lim_{(y_2 - y_1) \to 0} \Delta y \,, \quad \Delta z = \lim_{(z_2 - z_1) \to 0} \Delta z \,.$$

We can show differentials graphically as:



Differentials (infinitely small).

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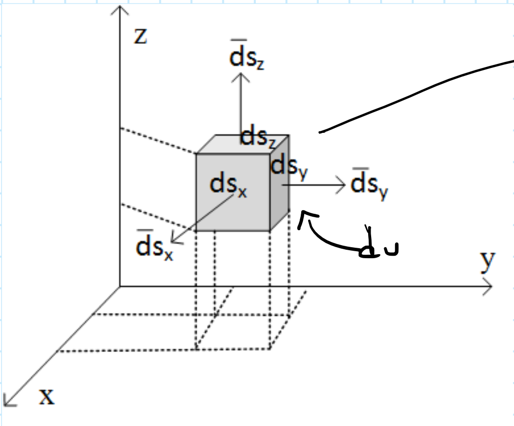
Differential areas: $ds_x = dy\,dz$, $ds_y = dx\,dz$, $ds_z = dx\,dy$

Graphically:



→ Infinitely small areas ⇒ Differential areas.

Vector differential areas:

$$\bar{ds}_x = \hat{a}_x \underbrace{dy\,dz}_{ds_x} = \hat{a}_x ds_x$$

$$\bar{ds}_y = \hat{a}_y\,dx\,dz \ , \quad \bar{ds}_z = \hat{a}_z\,dx\,dy$$

We have also **volume differential element**:

$$dv = dx\,dy\,dz \quad \text{(scaler)}$$

↳ only magnitude. no direction.

Vector Algebra:

1-) Vector Addition:

Given the following vectors:

$$\bar{v}_1 = \hat{a}_x v_{1x} + \hat{a}_y v_{1y} + \hat{a}_z v_{1z} ,$$
$$\bar{v}_2 = \hat{a}_x v_{2x} + \hat{a}_y v_{2y} + \hat{a}_z v_{2z} ,$$
$$\bar{v}_3 = \hat{a}_x v_{3x} + \hat{a}_y v_{3y} + \hat{a}_z v_{3z} .$$
$$\vdots$$

The sum of them can be written as:

$$\bar{v}_1 + \bar{v}_2 + \bar{v}_3 + \dots = \hat{a}_x (v_{1x} + v_{2x} + v_{3x} + \dots) + \hat{a}_y (v_{1y} + v_{2y} + v_{3y} + \dots) + \hat{a}_z (v_{1z} + v_{2z} + v_{3z} + \dots) .$$

If a vector is negative $(-\bar{v})$, it means the direction is reversed. $\xrightarrow{\quad}$ $\bar{v}$ $\quad \xleftarrow{\quad}$ $-\bar{v}$

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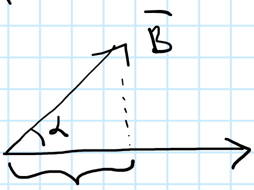
Thus, the subtraction of vectors can be written as addition:

For example $\bar{V}-\bar{V}_2 = \hat{a}_x(V_{1x}-V_{2x}) + \hat{a}_y(V_{1y}-V_{2y}) + \hat{a}_z(V_{1z}-V_{2z})$

2-) Vector Multiplication:

– Dot Product:

Let $\bar{A}$ and $\bar{B}$ be two vectors. The dot product of $\bar{A}$ and $\bar{B}$ is defined as:



$$\boxed{\bar{A}\cdot\bar{B} = |\bar{A}||\bar{B}|\cos(\alpha)}$$
(scalar)

, α = Angle between $\bar{A}$ and $\bar{B}$.

$|\bar{B}|\cos\alpha$ = Projection of $\bar{B}$ onto $\bar{A}$.

Remark: $\bar{A}\cdot\bar{B}$ is max. for when $\bar{A} \parallel \bar{B}$.
$\bar{A}\cdot\bar{B}$ is zero for when $\bar{A} \perp \bar{B}$.

In terms of components:

$$\bar{A} = \hat{a}_x A_x + \hat{a}_y A_y + \hat{a}_z A_z$$
$$\bar{B} = \hat{a}_x B_x + \hat{a}_y B_y + \hat{a}_z B_z$$

$$\boxed{\bar{A}\cdot\bar{B} = (A_x\cdot B_x) + (A_y B_y) + (A_z B_z) \quad \text{(scalar)}}$$

Properties:

Commutative law:
$\bar{A}\cdot\bar{B} = \bar{B}\cdot\bar{A}$

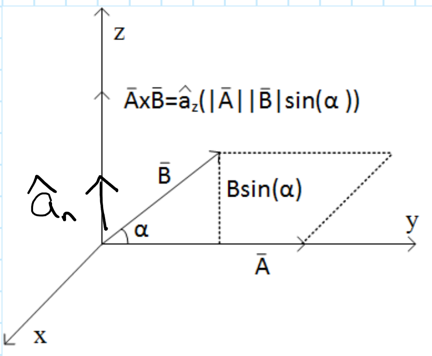
Associate law:
$\bar{A}\cdot(\bar{B}\cdot\bar{C}) = (\bar{A}\cdot\bar{B})\cdot\bar{C}$

Distributive law:
$\bar{A}\cdot(\bar{B}+\bar{C}) = \bar{A}\cdot\bar{B} + \bar{A}\cdot\bar{C}$

– Cross Product:

Let $\bar{A}$ and $\bar{B}$ be two vectors. The cross product of $\bar{A}$ and $\bar{B}$ is given as:

Graphically, $\bar{A}\times\bar{B} = \hat{a}_n |\bar{A}||\bar{B}|\sin\alpha$



Properties:

Commutative law:
$\bar{A}\times\bar{B} \neq \bar{B}\times\bar{A}$

Associate law:
$(\bar{A}\times\bar{B})\times\bar{C} \neq \bar{A}\times(\bar{B}\times\bar{C})$

Distributive law:
$\bar{A}\times(\bar{B}+\bar{C}) = \bar{A}\times\bar{B} + \bar{A}\times\bar{C}$

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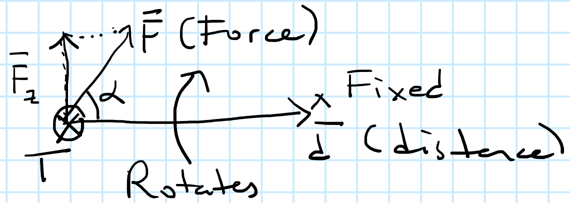
In terms of vector components, $\bar{A} = \hat{a}_x A_x + \hat{a}_y A_y + \hat{a}_z A_z$

$\bar{B} = \hat{a}_x B_x + \hat{a}_y B_y + \hat{a}_z B_z$.

$$\bar{A} \times \bar{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{a}_x (A_y B_z - A_z B_y) - \hat{a}_y (A_x B_z - A_z B_x) + \hat{a}_z (A_x B_y - A_y B_x)$$

Physical explanation:



$\bar{F} \times \bar{d} = $ Torque.

$\alpha = 90° \rightarrow$ max. torque. max. rotation

$\alpha = 0° \rightarrow T = 0$, no rotate.

Coordinate Systems:

There are 3 major coordinate systems we use in electromagnetic theory:

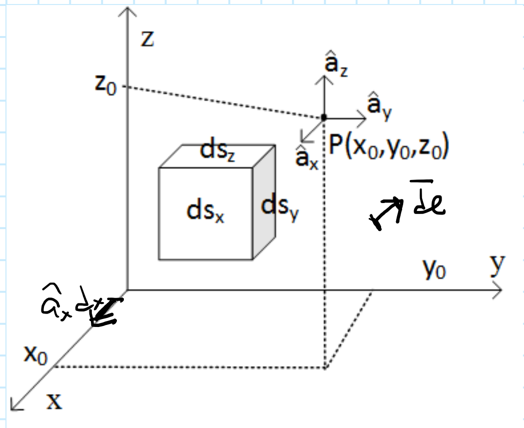
1-) Rectangular Coordinates (Cartesian Coordinates):

The unit vectors are: $\hat{a}_x, \hat{a}_y, \hat{a}_z$.

Axis: x, y, z.

A point P : $P(x, y, z)$

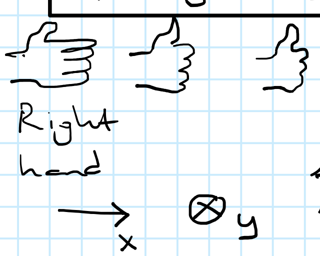
Dot products: $\boxed{\hat{a}_x \cdot \hat{a}_y = 0, \quad \hat{a}_x \cdot \hat{a}_z = 0, \quad \hat{a}_y \cdot \hat{a}_z = 0}$.

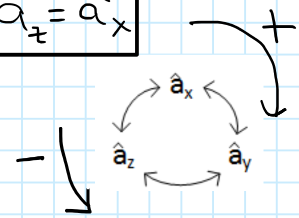


Rectangular coord. system.

Cross Products:

$\boxed{\hat{a}_x \times \hat{a}_y = \hat{a}_z, \quad \hat{a}_y \times \hat{a}_z = \hat{a}_x}$





$\hat{a}_z \times \hat{a}_x = \hat{a}_y$

$\boxed{\hat{a}_x \times \hat{a}_z = -\hat{a}_y}$ (ccw)

Differential length vector:

$$\boxed{\bar{dl} = \hat{a}_x dx + \hat{a}_y dy + \hat{a}_z dz}$$

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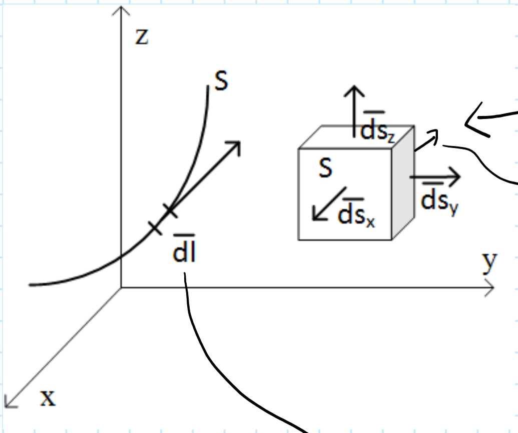
Differential area vectors: $\bar{ds}_x = \hat{a}_x\, dy\, dz$, $\bar{ds}_y = \hat{a}_y\, dx\, dz$, $\bar{ds}_z = \hat{a}_z\, dx\, dy$



$\bar{ds}_x = -\hat{a}_x\, dy\, dz$

The direction of $\bar{dl}$ is always tangential to s at the given point.

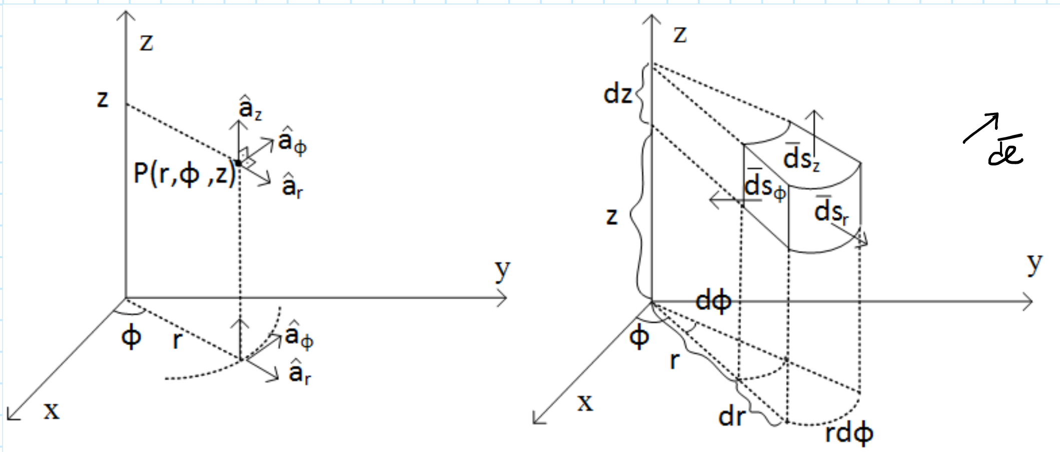
2-) Cylindrical Coordinate System:

Unit vectors: $\hat{a}_r$, $\hat{a}_\phi$, $\hat{a}_z$

Axis: r, ϕ, z

A given point P is shown by $P(r, \phi, z)$.

$$\boxed{\begin{aligned} \hat{a}_r \cdot \hat{a}_\phi &= 0 \\ \hat{a}_r \cdot \hat{a}_z &= 0 \\ \hat{a}_\phi \cdot \hat{a}_z &= 0 \end{aligned}}$$



$\hat{a}_r \to \hat{a}_\phi \to \hat{a}_z \to \hat{a}_r$

$$\boxed{\begin{aligned} \hat{a}_r \times \hat{a}_\phi &= \hat{a}_z \\ \hat{a}_\phi \times \hat{a}_z &= \hat{a}_r \\ \hat{a}_z \times \hat{a}_r &= \hat{a}_\phi \end{aligned}}$$

The differential legth vector:

$$\boxed{\bar{dl} = \hat{a}_r dr + \hat{a}_\phi r\, d\phi + \hat{a}_z dz}$$

The differential area vectors are:

$$\boxed{\begin{aligned} \bar{ds}_r &= \hat{a}_r\, r\, d\phi\, dz \\ \bar{ds}_\phi &= \hat{a}_\phi\, dr\, dz \\ \bar{ds}_z &= \hat{a}_z\, r\, dr\, d\phi \end{aligned}}$$

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Transformation Equations:

Point transformation:

We want to find $P(r, \phi, z)$ given the point P in cartesian coordinate system $P(x, y, z)$.

$$\boxed{r^2 = x^2 + y^2},$$

$$\boxed{\phi = \tan^{-1}\frac{y}{x}}, \quad \boxed{z = z}$$

Transformation from cylindrical to cartesian coord:

$$\boxed{x = r\cos\phi}, \quad \boxed{y = r\sin\phi}, \quad \boxed{z = z}$$

Vector Transformation:

Given a vector $\bar{A} = \hat{a}_r A_r + \hat{a}_\phi A_\phi + \hat{a}_z A_z$, we want to find $\bar{A}$ in rectangular coord. system.

$\bar{A} = \hat{a}_x A_x + \hat{a}_y A_y + \hat{a}_z A_z$ (rectangular coord.)

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix}$$

For transformation from rectangular to cartesian,

$$\begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

3.) Spherical Coordinate System:

Unit vectors: $\hat{a}_R, \hat{a}_\theta, \hat{a}_\phi$

Axis variables: R, θ, ϕ.

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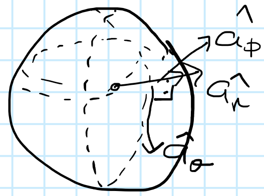
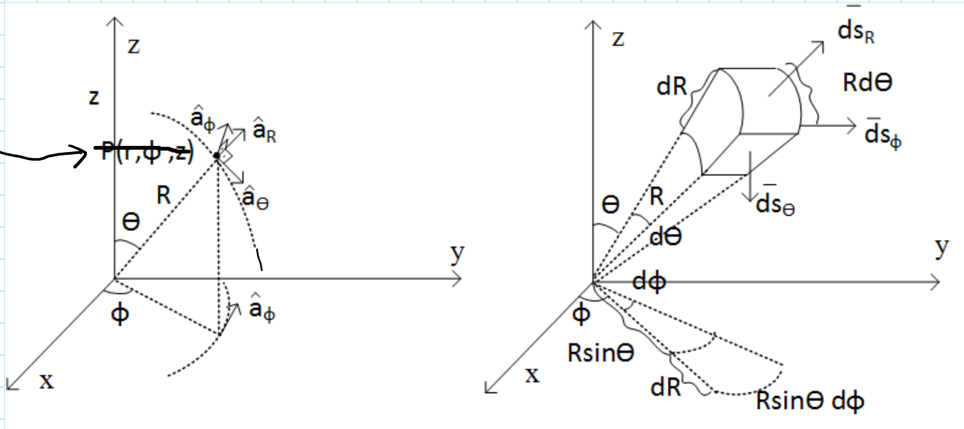
Given a point $P(R, \theta, \phi)$



$P(R,\theta,\varphi)$



Dot Products:

$\hat{a}_R \cdot \hat{a}_\theta = 0$

$\hat{a}_R \cdot \hat{a}_\varphi = 0$

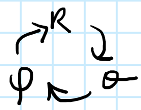
$\hat{a}_\theta \cdot \hat{a}_\varphi = 0$

Cross Products:

$\hat{a}_R \times \hat{a}_\theta = \hat{a}_\phi$

$\hat{a}_\theta \times \hat{a}_\varphi = \hat{a}_R$

$\hat{a}_\varphi \times \hat{a}_R = \hat{a}_\theta$



Vector Length Differential:

$$\overline{dl} = \hat{a}_R\, dR + \hat{a}_\theta\, R\, d\theta + \hat{a}_\varphi\, R\sin\theta\, d\phi$$

Vector Surface Differentials:

$$\overline{dS}_R = \hat{a}_R\, R^2 \sin\theta\, d\theta\, d\phi$$

$$\overline{dS}_\theta = \hat{a}_\theta\, R\sin\theta\, dR\, d\phi$$

$$\overline{dS}_\varphi = \hat{a}_\varphi\, R\, dR\, d\theta$$

Point Transformation:

Given a point in rectangular coordinates $P(x,y,z)$, we want to find $P(R,\theta,\varphi) = ?$

$$\boxed{R^2 = x^2 + y^2 + z^2}\ , \quad \boxed{\theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}}\ , \quad \boxed{\phi = \tan^{-1}\frac{y}{x}}\ .$$

From spherical to rectangular coord:

$$\boxed{\begin{aligned} x &= R\sin\theta\cos\phi \\ y &= R\sin\theta\sin\phi \\ z &= R\cos\theta \end{aligned}}$$

Vector Transformations:

Given a vector $\bar{A} = \hat{a}_R A_R + \hat{a}_\theta A_\theta + \hat{a}_\varphi A_\varphi$ in spherical coord.

We want $\bar{A} = \hat{a}_x A_x + \hat{a}_y A_y + \hat{a}_z A_z = ?$

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$\varphi = \phi$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\phi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_R \\ A_\theta \\ A_\varphi \end{bmatrix}$$

and

$$\begin{bmatrix} A_R \\ A_\theta \\ A_\varphi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}.$$

Vector Field:

is a group of vectors in a given space.
In 2D space, there is a vector defined at every point $P(x,y)$.
Then, this becomes a "vector field" in 2D space.
Its notation is

$$\bar{F}(x,y) = P(x,y)\,\hat{a}_x + Q(x,y)\,\hat{a}_y \qquad (2D).$$

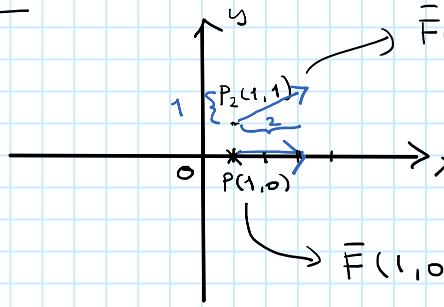
In 3D,

$$\bar{F}(x,y,z) = \underbrace{P(x,y,z)}_{\text{Functions}}\,\hat{a}_x + \underbrace{Q(x,y,z)}\,\hat{a}_y + \underbrace{T(x,y,z)}\,\hat{a}_z$$

Ex:

Draw some of the vectors of a vector field
$\bar{F}(x,y) = 2x\,\hat{a}_x + y\,\hat{a}_y$.

Ans:



$\bar{F}(1,1) = \bar{F} = 2\hat{a}_x + \hat{a}_y$

$\bar{F}(1,0) = 2\hat{a}_x$

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Integrals of Vector Fields:

There are 2 types of integrals of vector fields used in electromagnetic (E.M.) theory.

1-) Line integral (Curve integral)

Scalar ✗ Vector ✓

2-) Surface integrals

Scalar Vector ✓

1-) Line Integral of Vector Fields:

Suppose that C is a curve given in 3D space. Also, suppose that there exists a vector field $\bar{F}$ everywhere in this space.

Let's take a point P on this curve. Suppose $\bar{F}$ is at P.

$\bar{F}_t$ is the tangential component of $\bar{F}$ at P.

Then, $\bar{F}_t \cdot \bar{dl} = |F_t| \cdot |dl| = ?$

(Differential length vector is always tangential to the curve at point P.)

We can see that $dW = \bar{F} \cdot \bar{dl}$ (differential work) assuming $\bar{F}$ is a force.

So, we sum all dW's at every point on C from point a to b.

$$\Rightarrow \quad \text{Work} = \lim_{\Delta l \to 0} \sum_a^b \bar{F} \cdot \Delta \bar{l} = \int_a^b \bar{F} \cdot \bar{dl}$$

or

$$\text{Work} = \int_a^b \bar{F}(x,y,z) \cdot \bar{dl} \quad ✓$$

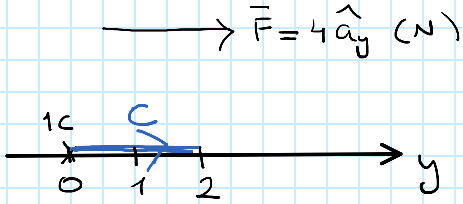
$$\boxed{\text{Work} = \int_a^b \bar{F} \cdot \bar{dl}}$$

(Line integral of a vector field $\bar{F}$ (force) gives us work)

Ex:

Given the force field $\bar{F} = 4\hat{a}_y$ (N). This force field moves a 1C of charge (proton or e^-) along the y-axis from $y=0$ to $y=2$m. Find the work done by this field?

Ans:



Solution Steps:

1-) Determine the proper coord. system: Rectangular.
Geometry = Curve. C.

2-) Use the formulas to find the unknown in the problem.

$$W = \int_a^b \bar{F} \cdot d\bar{l}$$

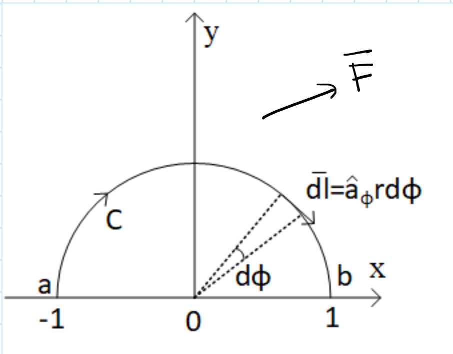
where $d\bar{l} = \hat{a}_x dx + \hat{a}_y dy + \hat{a}_z dz$

$$\Rightarrow \text{Work} = \int_a^b (4\hat{a}_y) \cdot (\hat{a}_x dx + \hat{a}_y dy + \hat{a}_z dz)$$

$$= \int_0^2 4\,dy = 4\int_0^2 dy = 4\, y\Big|_0^2 = 4\cdot(2-0) = 4\cdot 2 = 8 \text{ Joules.}$$

Ex:

Given $\bar{F}(x,y) = x\hat{a}_x + x\hat{a}_y$, the contour C is given as:



Find the work done in moving a 1C of charge from point a to point b along the contour C.

Ans:

1-) Coord. system? Geometry of the curve.
Cylindrical coord. is a proper choice for this problem.

2-) $W = \int_a^b \bar{F} \cdot d\bar{l}$ where $d\bar{l} = \cancel{\hat{a}_r dr} + \hat{a}_\phi r\,d\phi + \cancel{\hat{a}_z dz}$ (cylind. coord.)

and $\bar{F} = x\hat{a}_x + x\hat{a}_y$ (in rectangular coord.)

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We need to change $\bar{F}$ into cylindrical coord. First, let us transform the coord. variables:

$$\bar{F} = x\,\hat{a}_x + x\,\hat{a}_y = \underbrace{r\cos\varphi}_{F_x}\,\hat{a}_x + \underbrace{r\cos\varphi}_{F_y}\,\hat{a}_y \quad (N)$$

coord. variables.

We need to also change the vector into cartesian coord:

$$\begin{bmatrix} F_r \\ F_\phi \\ F_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r\cos\varphi \\ r\cos\varphi \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad F_r = r\cos^2\varphi + r\cos\varphi\sin\varphi$$
$$F_\varphi = -r\sin\varphi\cos\varphi + r\cos^2\varphi$$
$$F_z = 0.$$

$$\Rightarrow \quad \bar{F} = \hat{a}_r F_r + \hat{a}_\phi F_\phi + \hat{a}_z F_z = \hat{a}_r(r\cos^2\varphi + r\cos\varphi\sin\varphi) + \hat{a}_\varphi(-r\sin\varphi\cos\varphi + r\cos^2\varphi)$$

(the same $\bar{F}$ in cylindrical coord.)

Now, we can substitute $\bar{F}$ and $\bar{dl}$ into the formula:

$$W = \int_a^b (\hat{a}_r F_r + \hat{a}_\phi F_\phi)\cdot(\hat{a}_\phi\, r\,d\phi)$$

$$= \int_a^b F_\varphi \cdot r\,d\phi = \int_{-\pi}^{0} (-r\sin\phi\cos\varphi + r\cos^2\varphi)\,d\phi$$

where $r=1$ on C.

$$W = \int_{-\pi}^{0} (-\sin\phi\cos\varphi + \cos^2\varphi)\,d\varphi = \frac{\pi}{2}.$$

by using Matlab.

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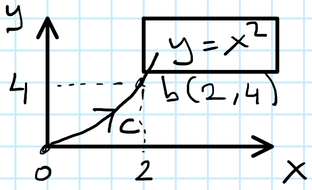
Ex:

Given $\bar{F} = \hat{a}_x x + \hat{a}_y xy$ as a force field.

and the curve is defined as:

C:



a(0,0)

Find the work done for moving a 1C of charge from point a to point b.

Ans:

$$W = \int_a^b \bar{F} \cdot \bar{dl} \quad \text{where } \bar{dl} = \hat{a}_x dx + \hat{a}_y dy$$

$$W = \int_a^b (\hat{a}_x x + \hat{a}_y xy) \cdot (\hat{a}_x dx + \hat{a}_y dy)$$

$$W = \int_a^b x\,dx + xy\,dy = \int_a^b x\,dx + \int_a^b xy\,dy$$

The curve is given by $y = x^2$

$$\frac{dy}{dx} = 2x \Rightarrow dy = 2x\,dx$$

$$\Rightarrow W = \int_0^2 x\,dx + \int_0^2 x(x^2)\,2x\,dx$$

Then,

$$W = \frac{1}{2}x^2\Big|_0^2 + 2\int_0^2 x^4\,dx = \frac{1}{2}(4) + 2\left(\frac{1}{5}x^5\right)\Big|_0^2$$

$$= 2 + \frac{2}{5}(2^5) = 2 + \frac{64}{5} = 14.8 \text{ J}.$$

2-) Surface Integral of Vector Fields:

Let us consider a surface S in a 3D space as shown below:

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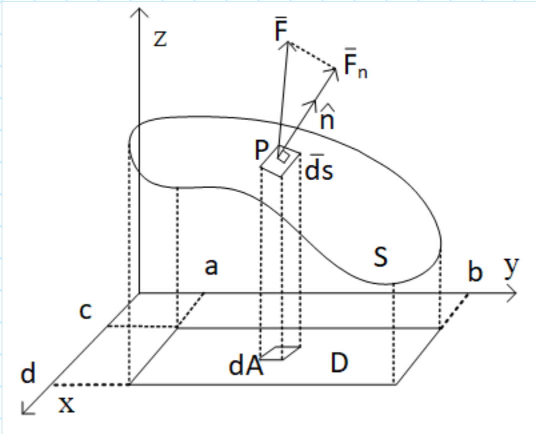


Suppose that there exists a vector field (force field) at every point in this space. Thus, $\bar{F}$ also exists at every point on S.

Consider a point P on S. We define $\bar{F}_n$ as the component of F normal to the surface at P.

We take a vector differential surface at point P as $\bar{ds}$. If we multiply $\bar{F}_n \cdot \bar{ds}$ and sum this operation over S, we obtain the "surface integral of vector fields".

Thus,

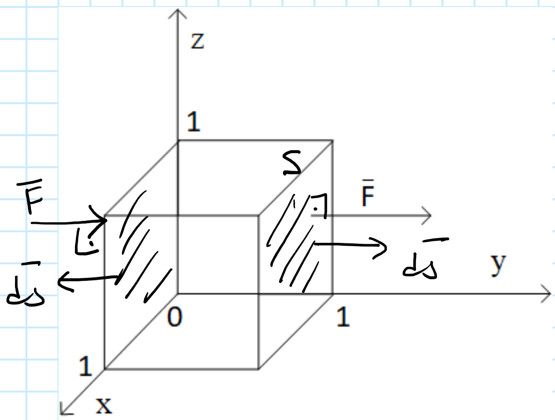
$$\text{Surface integral of vector field } \bar{F} = \iint_S \bar{F} \cdot \bar{ds}$$

We can also write this formula as:

$$\boxed{\text{Flux} = \int_S \bar{F} \cdot \bar{ds}}$$

Ex:

Given $\bar{F} = x^2\hat{a}_x + xy\hat{a}_y + yz\hat{a}_z$ (N), find the total flux that this force field creates on the surface of a unit cube located at the origin as shown below.



If $\bar{F} = 4a_y$ (N), then the solution is:

On the left side:

$\bar{ds} = -\hat{a}_y\, dx\, dz$

$$\text{Flux}_1 = \iint (4\hat{a}_y) \cdot (-\hat{a}_y\, dx\, dz) = \int_0^1 \int_0^1 4\, dx\, dz = -4.$$

On the right:

$\text{Flux}_2 = +4 \qquad \text{Flux} = \text{Flux}_1 + \text{Flux}_2 = 0$

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Ans:

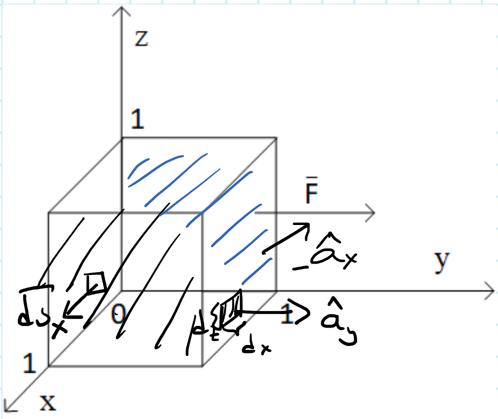
1-) Use rectangular coord because of the cube.

2-) $\text{Flux} = \int_S \bar{F} \cdot d\bar{s}$

$S : S_1 + S_2 + \cdots + S_6$

For each surface, we will find the flux and at the end sum the fluxes to find the total flux.

$\Rightarrow$ On the front face: $d\bar{s}_x = \hat{a}_x \, dy \, dz$, $x = 1$



$\text{Flux}_1 = \int\int \bar{F} \cdot d\bar{s}_x$

$= \int\int (x^2 \hat{a}_x) \cdot (\hat{a}_x \, dy \, dz)$

$\phi = \int_0^1 \int_0^1 x^2 \, dy \, dz = 1$

$x = 1$

For the back surface: $d\bar{s}_x = -\hat{a}_x \, dy \, dz$, $x = 0$,

$\text{Flux}_2 = \int_0^1 \int_0^1 -x^2 \, dy \, dz = 0$ //.

On the right surface: $d\bar{s}_y = \hat{a}_y \, dx \, dz$, $y = 1$,

$\text{Flux}_3 = \int_0^1 \int_0^1 xy \, dx \, dz = \int_0^1 \int_0^1 x \, dx \, dz = \int_0^1 \left(\frac{1}{2} x^2 \Big|_0^1 \right) dz$,

$= \frac{1}{2} \int_0^1 dz = \frac{1}{2}$ //

On the left surface: $d\bar{s}_y = -\hat{a}_y \, dx \, dz$, $y = 0$,

$\text{Flux}_4 = \int_0^1 \int_0^1 -xy \, dx \, dz = 0.$

On the top surface: $d\bar{s}_z = \hat{a}_z \, dx \, dy$, $z = 1$,

$\text{Flux}_5 = \int_0^1 \int_0^1 yz \, dx \, dy = \frac{1}{2}.$

On the bottom surface: $d\bar{s}_z = -\hat{a}_z \, dx \, dy$, $z = 0$

$\text{Flux}_6 = \int_0^1 \int_0^1 -yz \, dx \, dy = 0.$

$\text{Flux}_{total} = \text{Flux}_1 + \text{Flux}_2 + \cdots + \text{Flux}_6 = 2$ //

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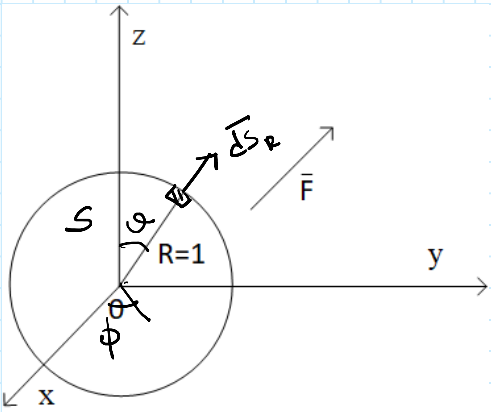
Ex:

$\bar{F} = \hat{a}_R R^2$ (N), find the total flux that this force field creates on a unit sphere?

Ans:

Let us draw the geometry of the problem:



1-) Spherical coord. system because the surface is a sphere.

2-) $\text{Flux} = \iint_S \bar{F} \cdot \bar{ds}$

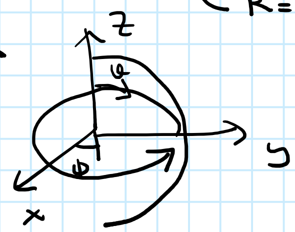
where

$\bar{ds} = \bar{ds}_R = \hat{a}_R R^2 \sin\theta \, d\theta \, d\phi$

$\text{Flux} = \iint R^4 \sin\theta \, d\theta \, d\phi$

R=1 on the sphere.

$\Rightarrow \text{Flux} = \int_0^{2\pi}\int_0^{\pi} \sin\theta \, d\theta \, d\phi$



$= -\int_0^{2\pi} (\cos\theta)\Big|_0^{\pi} d\phi$

$= 2\int_0^{2\pi} d\theta = 4\pi.$ //

Gradient of a Scalar Field:

Define the grad or del operator as:

$$\nabla = \text{Grad} = \text{Del} = \hat{a}_x \frac{\partial}{\partial x} + \hat{a}_y \frac{\partial}{\partial y} + \hat{a}_z \frac{\partial}{\partial z}$$

Let $f(x,y,z)$ be a scalar function, then

$$\nabla f(x,y,z) = \hat{a}_x \frac{\partial}{\partial x} f(x,y,z) + \hat{a}_y \frac{\partial}{\partial y} f(x,y,z) + \hat{a}_z \frac{\partial f(x,y,z)}{\partial z}$$

Thus, the gradient of a scalar function f gives us the rate of change vector in space coordinates (distance).

In general notation,

$$\boxed{\nabla f = \hat{a}_{u_1} \frac{\partial f}{h_1 \partial u_1} + \hat{a}_{u_2} \frac{\partial f}{h_2 \partial U_2} + \hat{a}_{u_3} \frac{\partial f}{h_3 \partial U_3}}$$

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where the parameters $u_1, u_2, u_3, \hat{a}_{u_1}, \hat{a}_{u_2}, \hat{a}_{u_3}$ and h_1, h_2, h_3 are given as:

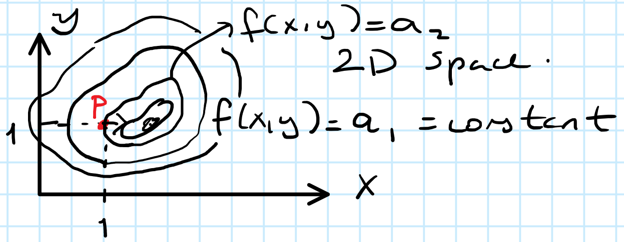
Koordinat Sistemi:	Kartezyen	Silindirik	Küresel
u_1	x	r	R
u_2	y	ϕ	θ
u_3	z	z	ϕ
h_1	1	1	1
h_2	1	r	R
h_3	1	1	$R sin\theta$

Table 1.1: Koordinat Sistemi Sabitleri

Ex:

Given a scalar function $f(x,y) = x^2+y^2$ find the gradient of this function at point $P(1,1)$?

(For instance, $f(x,y)$ may represent the height of a geographical area.)



Ans:

$$\vec{\nabla} f(x,y,z) = \vec{\nabla} f = \hat{a}_x \frac{\partial f}{\partial x} + \hat{a}_y \frac{\partial f}{\partial y} + \hat{a}_z \frac{\partial f}{\partial z}$$

$$= \hat{a}_x 2x + \hat{a}_y 2y$$

$$\vec{\nabla} f \Big|_{P(1,1)} = \hat{a}_x 2 + \hat{a}_y 2$$

What does it mean graphically?

Let us draw $\underbrace{f(x,y)}_{z} = x^2+y^2$ (3D graph that gives us a surface)

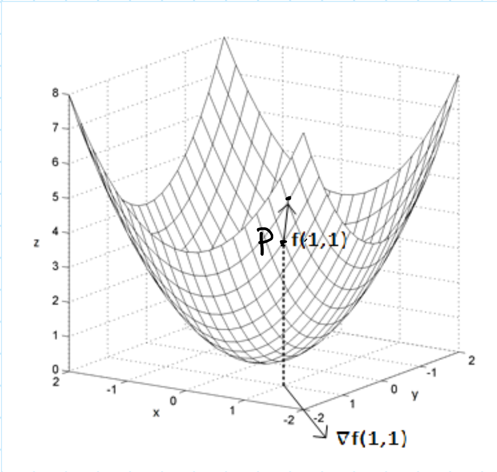
$$z = x^2 + y^2$$

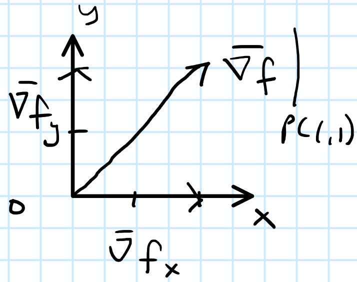
or

$$\underbrace{x^2+y^2-z}_{f(x,y,z)} = 0$$

$$f(x,y,z) = 0$$

The graph of $f(x,y) = x^2 + y^2$ is given as :





- Thus, the gradient of f at P gives the vector that shows the max. change of f in that direction.

Divergence of a vector field:

Let $\bar{F} = \hat{a}_x F_x + \hat{a}_y F_y + \hat{a}_z F_z$ be a vector field,

$$\mathrm{div.}\bar{F}(x,y,z) = \bar{\nabla}.\bar{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \quad \text{(scalar)}$$

In a general notation: for $\bar{F} = \hat{a}_{u_1} F_1 + \hat{a}_{u_2} F_2 + \hat{a}_{u_3} F_3$

$$\bar{\nabla}.\bar{F}(x,y,z) \leftarrow \boxed{\bar{\nabla}.\bar{F} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 F_1) + \frac{\partial}{\partial u_2}(h_1 h_3 F_2) + \frac{\partial}{\partial u_3}(h_1 h_2 F_3)\right]}$$

where $h_1, h_2, h_3, u_1, u_2, u_3$ are given in Table 1.1.

The divergence is also applicable at a point.

The divergence gives the rate of change of $\bar{F}$ when it leaves point P.

Ex:

Given the vector field $\bar{F}(x) = \hat{a}_x \underbrace{\frac{1}{2}x}_{F_x}$. Find $\bar{\nabla}.\bar{F}$ at point $P(1,1)$?

Ans:

$$\bar{\nabla}.\bar{F}\Big|_{P(1,1)} = \frac{\partial}{\partial x}F_x + \frac{\partial}{\partial y}\cancel{F_y}^{\,0} + \frac{\partial}{\partial z}\cancel{F_z}^{\,0} = \frac{\partial}{\partial x}\left(\frac{1}{2}x\right)\Big|_{P(1,1)} = \frac{1}{2} \,//$$

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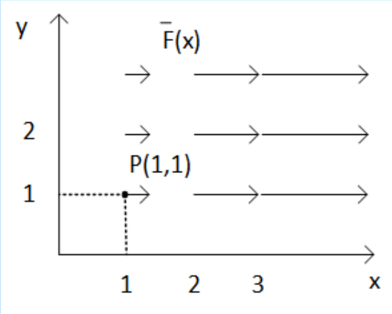
Graphical interpretation of the problem is as follows:

Let us draw the vector field $\bar{F} = \hat{a}_x \frac{1}{2} x$



– At point P(1,1) vector field $\bar{F}$ coming to P from left is not equal to the field leaving from right.

$$\bar{F}_i \longrightarrow \underset{dx}{\overset{P}{\bullet}} \longrightarrow \bar{F}_e \qquad \bar{F}_e > \bar{F}_i$$

– Therefore $\bar{\nabla} . \bar{F} > 0$ means there is a source at P. and the value of $\bar{\nabla} . \bar{F}$ gives us how much $\bar{F}$ diverges from the point P.

– $\bar{\nabla} . \bar{F} \big|_P = 0$, this means that there is no $\underbrace{\overbrace{\text{source}}^{\text{source free}}}_{\text{charge}}$ at point P.

– $\bar{\nabla} . \bar{F} < 0$ means there is a sink at point P.

Ex:

Magnetic field (vector field of force) created by a straight current carrying wire is given by

$$\bar{B} = \hat{a}_\phi \frac{k}{r} \left(\frac{Wb}{m^2}\right), \quad \text{where } k = \text{constant}.$$

$\hookrightarrow$ Mag. field vector. $\qquad$ r = radial distance from the wire.

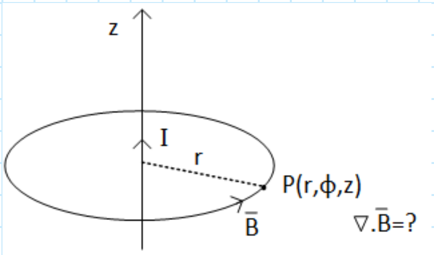
Find $\bar{\nabla} . \bar{B}(r) = ?$

$\hookrightarrow \bar{\nabla} . \bar{B} = ?$

Ans:

Step 1: Cylindrical coord.

Step 2:



$$\bar{\nabla} . \bar{B} = \frac{1}{r}\left[\frac{\partial}{\partial r}(r \cancel{B_r})^{0} + \frac{\partial}{\partial \phi}(B_\phi) + \frac{\partial}{\partial z}(r \cancel{B_z})^{0}\right]$$

$$\Rightarrow \bar{\nabla} . \bar{B} = \frac{1}{r} \frac{\partial}{\partial \phi}(B_\phi)$$

$$= \frac{1}{r} \frac{\partial}{\partial \phi}\left(\frac{k}{r}\right) = 0. \;// \text{ (source free)}$$

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Curl of a Vector Field:

Let $\bar{F} = \hat{a}_x F_x + \hat{a}_y F_y + \hat{a}_z F_z$ be a vector field

$$\Rightarrow \text{curl } \bar{F} = \bar{\nabla} \times \bar{F} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \hat{a}_x \left(\frac{\partial}{\partial y} F_z - \frac{\partial}{\partial z} F_y \right) - \hat{a}_y \left(\frac{\partial}{\partial x} F_z - \frac{\partial}{\partial z} F_x \right) + \hat{a}_z \left(\frac{\partial}{\partial x} F_y - \frac{\partial}{\partial y} F_x \right).$$

In general notation: $\bar{F} = \hat{a}_{u_1} F_1 + \hat{a}_{u_2} F_2 + \hat{a}_{u_3} F_3$

$$\boxed{\bar{\nabla} \times \bar{F} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \hat{a}_{u_1} h_1 & \hat{a}_{u_2} h_2 & \hat{a}_{u_3} h_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix}} = \hat{a}_{u_1} h_1 \left\{ \left[\frac{\partial}{\partial u_2} (h_3 F_3) - \frac{\partial}{\partial u_3} (h_2 F_2) \right] - \hat{a}_{u_2} h_2 \left[\frac{\partial}{\partial u_1} (h_3 F_3) - \frac{\partial}{\partial u_3} (h_1 F_1) \right] + \hat{a}_{u_3} h_3 \left[\frac{\partial}{\partial u_1} (h_2 F_2) - \frac{\partial}{\partial u_2} (h_1 F_1) \right] \right\}$$

Physical interpretation of the curl of a vector field: