EE 209 Lecture Notes:

P1

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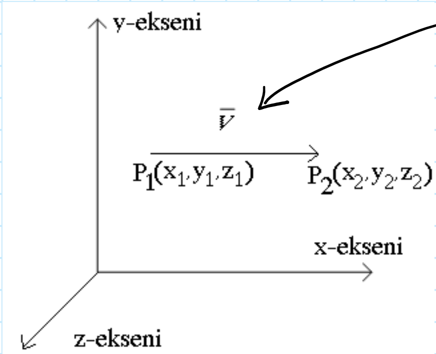
Vector Calculus:

Vectors:

A vector is a quantity with a magnitude and direction.

Electric and magnetic fields are vectors.

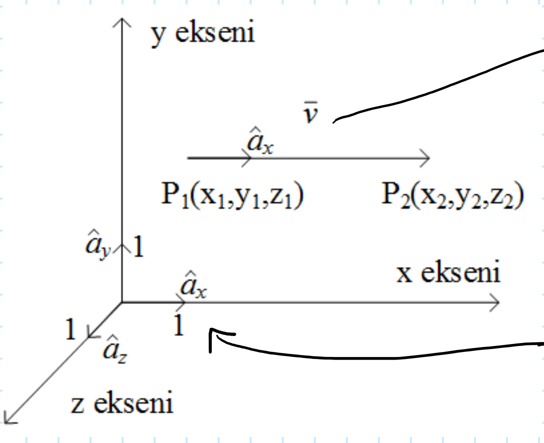


$\bar{v}$ is a vector.

Unit vector: It is a vector with unity magnitude

$\|\bar{a}\|$ = Norm of a vector = Magnitude of a vector

For unit vector $\|\bar{a}\| = |\bar{a}| = 1$.



$$\bar{v} = \hat{a}_x \|\bar{v}\|$$

direction — magnitude.

$$\|\bar{v}\| = \left[(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2\right]^{\frac{1}{2}}$$

unit vectors in the x, y and z directions.

The vector $\bar{v}$ in the above figure is directed on the x-axis. Thus,

$$\bar{v} = \hat{a}_x \cdot \underbrace{V_x}_{\text{magnitude along the x-direction}}$$

Any vector in 3D space can be composed by components.

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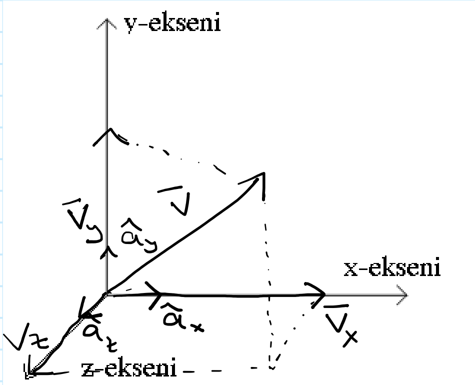


$$\bar{V} = \bar{V}_x + \bar{V}_y + \bar{V}_z$$

components.

where

$$\bar{V}_x = \hat{a}_x V_x \ , \quad \bar{V}_y = \hat{a}_y V_y \ , \quad \bar{V}_z = \hat{a}_z V_z$$

Thus,

$$\bar{V} = \hat{a}_x V_x + \hat{a}_y V_y + \hat{a}_z V_z \quad \checkmark$$

$$\bar{V} = \langle V_x , V_y , V_z \rangle$$ (notation used by mathematicians)

Differentials (scalar, vector):

In a 3D space, to show difference in distance we use the following notations:

$\Delta x = x_2 - x_1 =$ difference in distance in the x-direction.

$$\underset{0 \quad x_1 \quad x_2}{\xrightarrow{\quad \overset{\Delta x}{\longleftrightarrow} \quad}} x$$

Similarly,

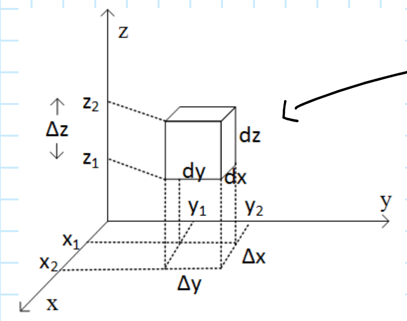
$$\Delta y = y_2 - y_1 \ , \quad \Delta z = z_2 - z_1 \ .$$

Define scalar length differentials as:

$$dx = \lim_{(x_2 - x_1) \to 0} \Delta x \ ,$$

$$dy = \lim_{(y_2 - y_1) \to 0} \Delta y \ , \quad dz = \lim_{(z_2 - z_1) \to 0} \Delta z \ .$$

We can show the length differentials graphically as:

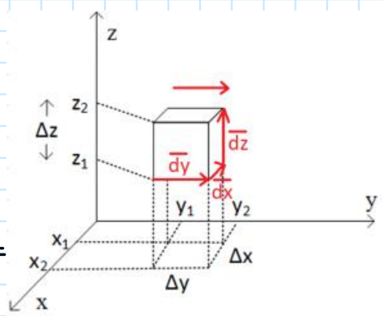


Differentials

We can also define

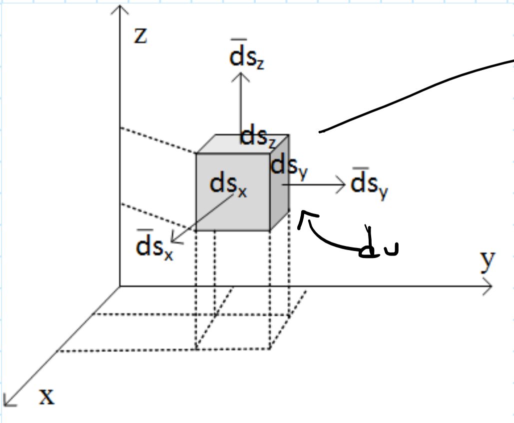
Vector Length Differentials:

$$\bar{dx} = \hat{a}_x dx \ , \quad \bar{dy} = \hat{a}_y dy \ , \quad \bar{dz} = \hat{a}_z dz$$



Differential areas: $ds_x = dy\,dz$, $ds_y = dx\,dz$, $ds_z = dx\,dy$

Graphically:



Infinitely small areas = Differential areas.

Vector differential areas:

$$\bar{ds}_x = \hat{a}_x \underbrace{dy\,dz}_{ds_x} = \hat{a}_x ds_x$$

$$\bar{ds}_y = \hat{a}_y\, dx\,dz\,,\quad \bar{ds}_z = \hat{a}_z\, dx\,dy$$

We have also volume differential element:

$$du = dx\,dy\,dz \quad (\text{scalar})$$

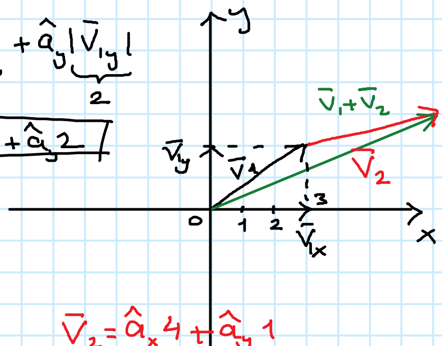
↳ only magnitude. no direction.

Vector Algebra:

1-) Vector Addition:

$\bar{V}_1 = \bar{V}_{1x} + \bar{V}_{1y} = \hat{a}_x \underbrace{|\bar{V}_{1x}|}_{3} + \hat{a}_y \underbrace{|\bar{V}_{1y}|}_{2}$

Given the following vectors: $= \boxed{\hat{a}_x 3 + \hat{a}_y 2}$



$$\bar{v}_1 = \hat{a}_x v_{1x} + \hat{a}_y v_{1y} + \hat{a}_z v_{1z}\,,$$
$$\bar{V}_2 = \hat{a}_x v_{2x} + \hat{a}_y v_{2y} + \hat{a}_z v_{2z}\,,$$
$$\bar{V}_3 = \hat{a}_x v_{3x} + \hat{a}_y v_{3y} + \hat{a}_z v_{3z}\,.$$
$$\vdots$$

$\bar{V}_2 = \hat{a}_x 4 + \hat{a}_y 1$

$\bar{V}_1 + \bar{V}_2 = \hat{a}_x(3+4) + \hat{a}_y(2+1)$
$= \underline{\hat{a}_x 7 + \hat{a}_y 3}$

The sum of them can be written as:

$$\bar{v}_1 + \bar{v}_2 + \bar{v}_3 + \dots = \hat{a}_x(v_{1x} + v_{2x} + v_{3x} + \dots) + \hat{a}_y(v_{1y} + v_{2y} + v_{3y} + \dots) + \hat{a}_z(v_{1z} + v_{2z} + v_{3z} + \dots).$$

If a vector is negative $(-\bar{v})$, it means the direction is reversed. $\xrightarrow[\bar{v}]{}$ $\xleftarrow[-\bar{v}]{}$

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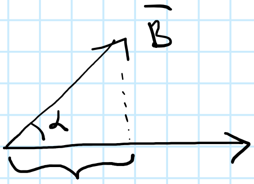
Thus, the subtraction of vectors can be written as addition:

For example $\bar{v}_1 - \bar{v}_2 = \hat{a}_x(v_{1x} - v_{2x}) + \hat{a}_y(v_{1y} - v_{2y}) + \hat{a}_z(v_{1z} - v_{2z})$

2-) Vector Multiplication:

– Dot Product:

Let $\bar{A}$ and $\bar{B}$ be two vectors. The dot product of $\bar{A}$ and $\bar{B}$ is defined as:



$$\boxed{\bar{A}\cdot\bar{B} = |\bar{A}||\bar{B}|\cos(\alpha)}$$
(scalar)

, α = Angle between $\bar{A}$ and $\bar{B}$.

$|\bar{B}|\cos\alpha$ = Projection of $\bar{B}$ onto $\bar{A}$.

Remark: $\bar{A}\cdot\bar{B}$ is max. for when $\bar{A} \parallel \bar{B}$.
$\bar{A}\cdot\bar{B}$ is zero for when $\bar{A} \perp \bar{B}$.

In terms of components:

$$\bar{A} = \hat{a}_x A_x + \hat{a}_y A_y + \hat{a}_z A_z$$
$$\bar{B} = \hat{a}_x B_x + \hat{a}_y B_y + \hat{a}_z B_z$$

$$\boxed{\bar{A}\cdot\bar{B} = (A_x \cdot B_x) + (A_y B_y) + (A_z B_z) \quad \text{(scalar)}}$$

Properties:

Commutative law:
$\bar{A}\cdot\bar{B} = \bar{B}\cdot\bar{A}$

Associate law:
$\bar{A}\cdot(\bar{B}\cdot\bar{C}) = (\bar{A}\cdot\bar{B})\cdot\bar{C}$

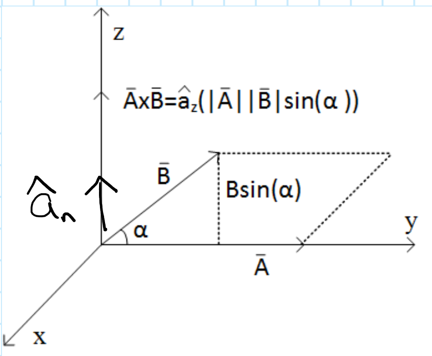
Distributive law:
$\bar{A}\cdot(\bar{B}+\bar{C}) = \bar{A}\cdot\bar{B} + \bar{A}\cdot\bar{C}$

– Cross Product:

Let $\bar{A}$ and $\bar{B}$ be two vectors. The cross product of $\bar{A}$ and $\bar{B}$ is given as:

$$\boxed{\bar{A}\times\bar{B} = \hat{a}_n |\bar{A}||\bar{B}|\sin\alpha}$$

Graphically,



Note:
$\bar{B}$ is on the xy-plane

Properties:

Commutative law:
$\bar{A}\times\bar{B} \neq \bar{B}\times\bar{A}$

Associate law:
$(\bar{A}\times\bar{B})\times\bar{C} \neq \bar{A}\times(\bar{B}\times\bar{C})$

Distributive law:
$\bar{A}\times(\bar{B}+\bar{C}) = \bar{A}\times\bar{B} + \bar{A}\times\bar{C}$

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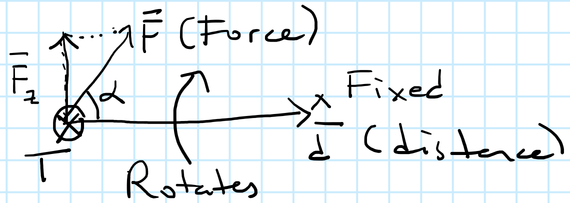
In terms of vector components, $\bar{A} = \hat{a}_x A_x + \hat{a}_y A_y + \hat{a}_z A_z$
$\bar{B} = \hat{a}_x B_x + \hat{a}_y B_y + \hat{a}_z B_z$.

$$\bar{A} \times \bar{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{a}_x (A_y B_z - A_z B_y) - \hat{a}_y (A_x B_z - A_z B_x) + \hat{a}_z (A_x B_y - A_y B_x)$$

Physical explanation:



$\bar{F} \times \bar{d} =$ Torque.

$\alpha = 90°$ → max. torque.
max. rotation

$\alpha = 0°$ → $T = 0$, no rotate.

Coordinate Systems:

There are 3 major coordinate systems we use in electromagnetic theory:

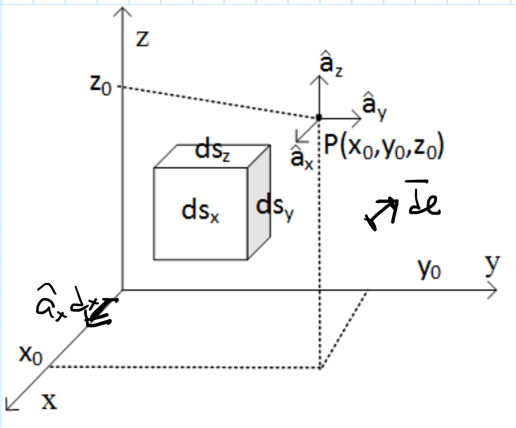
1-) Rectangular Coordinates (Cartesian Coordinates):

The unit vectors are: $\hat{a}_x, \hat{a}_y, \hat{a}_z$.

Axis: x, y, z.

A point P : $P(x,y,z)$

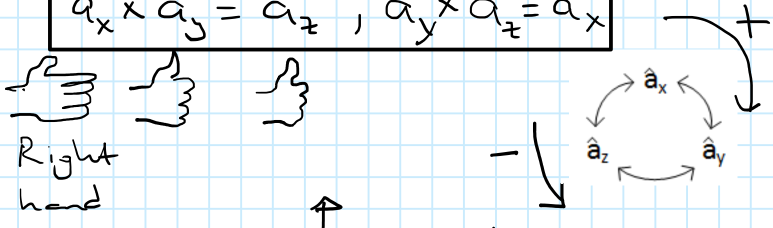
Dot products: $\boxed{\hat{a}_x \cdot \hat{a}_y = 0, \quad \hat{a}_x \cdot \hat{a}_z = 0, \quad \hat{a}_y \cdot \hat{a}_z = 0}$.



Rectangular coord. system.

Cross Products:

$\boxed{\hat{a}_x \times \hat{a}_y = \hat{a}_z, \quad \hat{a}_y \times \hat{a}_z = \hat{a}_x}$



x ⊗y

$\hat{a}_z \times \hat{a}_x = \hat{a}_y$

$\boxed{\hat{a}_x \times \hat{a}_z = -\hat{a}_y}$ (ccw)

Differential length vector:

$$\boxed{\bar{dl} = \hat{a}_x dx + \hat{a}_y dy + \hat{a}_z dz}$$

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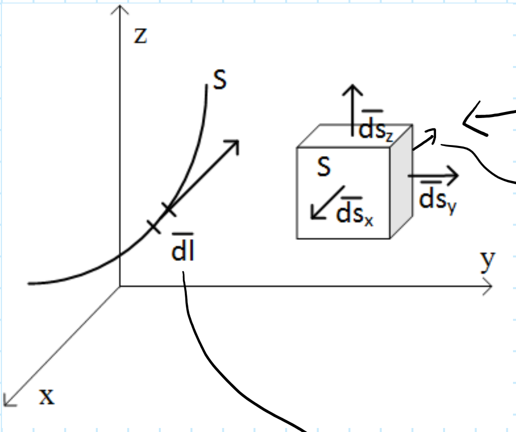
Differential area vectors: $d\bar{S}_x = \hat{a}_x\,dy\,dz$, $d\bar{S}_y = \hat{a}_y\,dx\,dz$, $d\bar{S}_z = \hat{a}_z\,dx\,dy$



$d\bar{S}_x = -\hat{a}_x\,dy\,dz$

The direction of $d\bar{l}$ is always tangential to S at the given point.

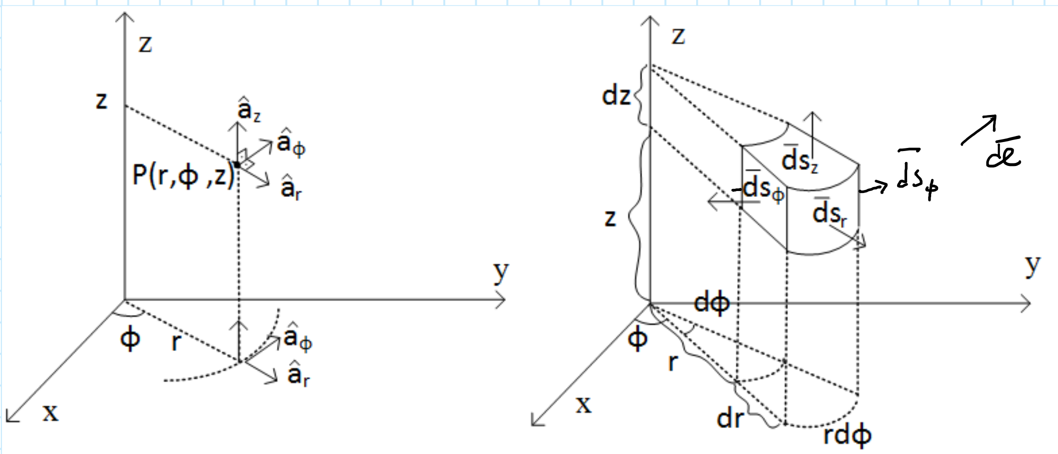
2-) Cylindrical Coordinate System:

Unit vectors: $\hat{a}_r$, $\hat{a}_\phi$, $\hat{a}_z$

Coordinate variables: r, ϕ, z

A given point P is shown by $P(r, \phi, z)$.

$$\boxed{\begin{aligned} \hat{a}_r \cdot \hat{a}_\phi &= 0 \\ \hat{a}_r \cdot \hat{a}_z &= 0 \\ \hat{a}_\phi \cdot \hat{a}_z &= 0 \end{aligned}}$$



$\hat{a}_r \to \hat{a}_\phi \to \hat{a}_z \to \hat{a}_r$

$$\boxed{\begin{aligned} \hat{a}_r \times \hat{a}_\phi &= \hat{a}_z \\ \hat{a}_\phi \times \hat{a}_z &= \hat{a}_r \\ \hat{a}_z \times \hat{a}_r &= \hat{a}_\phi \end{aligned}}$$

The differential length vector:

$$\boxed{d\bar{l} = \hat{a}_r\,dr + \hat{a}_\phi\,r\,d\phi + \hat{a}_z\,dz}$$

The differential area vectors are:

$$\boxed{\begin{aligned} d\bar{S}_r &= \hat{a}_r\,r\,d\phi\,dz \\ d\bar{S}_\phi &= \hat{a}_\phi\,dr\,dz \\ d\bar{S}_z &= \hat{a}_z\,r\,dr\,d\phi \end{aligned}}$$

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Transformation Equations:

Point transformation:

We want to find $P(r, \phi, z)$ given the point P in cartesian coordinate system $P(x, y, z)$.

$\boxed{r^2 = x^2 + y^2}$,

$\boxed{\phi = \tan^{-1}\frac{y}{x}}$, $\boxed{z = z}$

Transformation from cylindrical to cartesian coord:

$\boxed{x = r\cos\phi}$, $\boxed{y = r\sin\phi}$, $\boxed{z = z}$

Vector Transformation:

Given a vector $\bar{A} = \hat{a}_r A_r + \hat{a}_\phi A_\phi + \hat{a}_z A_z$, we want to find $\bar{A}$ in rectangular coord. system.

$\bar{A} = \hat{a}_x A_x + \hat{a}_y A_y + \hat{a}_z A_z$ (rectangular coord.)

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix}, \quad \begin{aligned} A_x &= A_r\cos\phi - A_\phi\sin\phi \\ A_y &= A_r\sin\phi + A_\phi\cos\phi \\ A_z &= A_z \end{aligned}$$

For transformation from rectangular to cylindrical., components.

$$\begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}, \quad \begin{aligned} A_r &= A_x\cos\phi + A_y\sin\phi \\ A_\phi &= -A_x\sin\phi + A_y\cos\phi \\ A_z &= A_z. \end{aligned}$$

3.) Spherical Coordinate System:

Unit vectors: $\hat{a}_R, \hat{a}_\theta, \hat{a}_\phi$

Axis variables: R, θ, ϕ.

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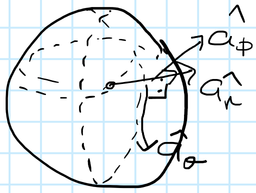
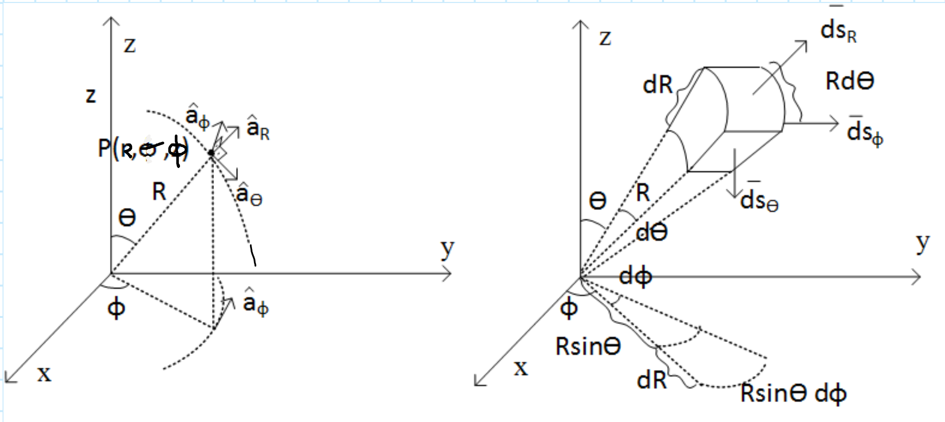


Given a point $P(R, \theta, \phi)$



Dot Products:

$\hat{a}_R \cdot \hat{a}_\theta = 0$

$\hat{a}_R \cdot \hat{a}_\varphi = 0$

$\hat{a}_\theta \cdot \hat{a}_\varphi = 0$

Cross Products:

$\hat{a}_R \times \hat{a}_\theta = \hat{a}_\phi$

$\hat{a}_\theta \times \hat{a}_\varphi = \hat{a}_R$

$\hat{a}_\varphi \times \hat{a}_R = \hat{a}_\theta$

$R \to \theta \to \varphi \to R$

Vector Length Differential:

$$\bar{dl} = \hat{a}_R\, dR + \hat{a}_\theta\, R\, d\theta + \hat{a}_\varphi\, R \sin\theta\, d\phi$$

Vector Surface Differentials:

$$\bar{ds}_R = \hat{a}_R\, R^2 \sin\theta\, d\theta\, d\phi$$

$$\bar{ds}_\theta = \hat{a}_\theta\, R \sin\theta\, dR\, d\phi$$

$$\bar{ds}_\varphi = \hat{a}_\varphi\, R\, dR\, d\theta$$

Point Transformation:

Given a point in rectangular coordinates $P(x, y, z)$, we want to find $P(R, \theta, \varphi) = ?$

$$\boxed{R^2 = x^2 + y^2 + z^2}\ , \quad \boxed{\theta = \tan^{-1} \frac{\sqrt{x^2+y^2}}{z}}\ , \quad \boxed{\phi = \tan^{-1} \frac{y}{x}}\ .$$

From spherical to rectangular coord:

$$\boxed{\begin{aligned} x &= R \sin\theta \cos\phi \\ y &= R \sin\theta \sin\phi \\ z &= R \cos\theta \end{aligned}}$$

Vector Transformations:

Given a vector $\bar{A} = \hat{a}_R A_R + \hat{a}_\theta A_\theta + \hat{a}_\varphi A_\varphi$ in spherical coord.

We want $\bar{A} = \hat{a}_x A_x + \hat{a}_y A_y + \hat{a}_z A_z = ?$

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Vector Field:

is a group of vectors in a given space.
In 2D space, there is a vector defined at every point $P(x,y)$.
Then, this becomes a "vector field" in 2D space.
Its notation is

$$\bar{F}(x,y) = P(x,y)\,\hat{a}_x + Q(x,y)\,\hat{a}_y \qquad (2D).$$

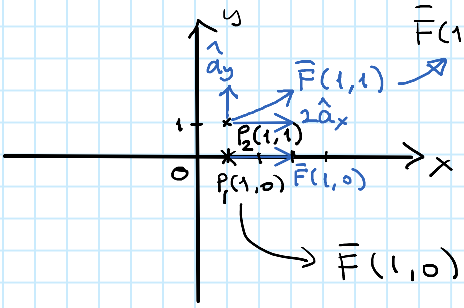
In a 3D-space

$$\bar{F}(x,y,z) = \underbrace{P(x,y,z)}\,\hat{a}_x + \underbrace{Q(x,y,z)}\,\hat{a}_y + \underbrace{T(x,y,z)}_{\text{Functions}}\,\hat{a}_z$$

Ex:

Draw some of the vectors of a vector field
$\bar{F}(x,y) = 2x\,\hat{a}_x + y\,\hat{a}_y$.

Ans:



$\bar{F}(1,1) = \bar{F} = 2\hat{a}_x + \hat{a}_y$

$\bar{F}(1,0) = 2\hat{a}_x$

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Wednesday, October 12, 2022 11:19 AM

Integrals of Vector Fields:

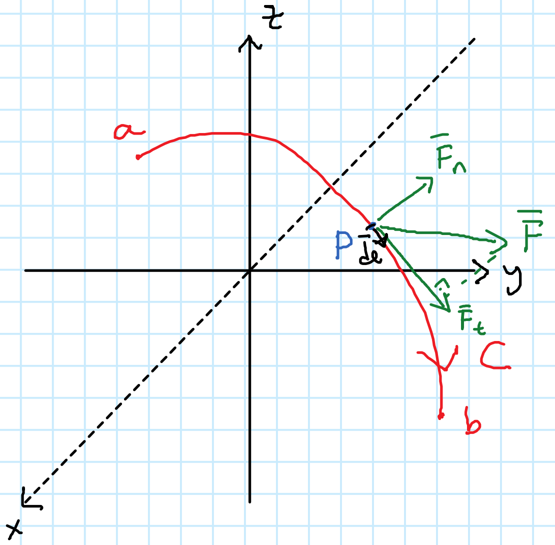
There are 2 types of integrals of vector fields used in electromagnetic (E.M.) theory.

1-) Line integral (Curve integral)
 - Scalar ✗
 - Vector ✓

2-) Surface integrals
 - Scalar
 - Vector ✓

1-) Line Integral of Vector Fields:

Suppose that C is a curve given in 3D space. Also, suppose that there exists a vector field $\bar{F}$ everywhere in this space.



Let's take a point P on this curve. Suppose $\bar{F}$ is at P. $\bar{F}_t$ is the tangential component of $\bar{F}$ at P.

Then, $\bar{F}_t \cdot \bar{d\ell} = |F_t| \cdot |d\ell| = ?$

(Differential length vector is always tangential to the curve at point P.)

We can see that $dW = \bar{F} \cdot \bar{d\ell}$ (differential work) assuming $\bar{F}$ is a force.

So, we sum all dW's at every point on C from point a to b.

$$\Rightarrow \text{Work} = \lim_{\Delta\ell \to 0} \sum_a^b \bar{F} \cdot \bar{\Delta\ell} = \int_a^b \bar{F} \cdot \bar{d\ell}$$

or

$$\text{Work} = \int_a^b \bar{F}(x,y,z) \cdot \bar{d\ell} \quad ✓$$

$$\boxed{\text{Work} = \int_a^b \bar{F} \cdot \bar{d\ell}}$$

(Line integral of a vector field $\bar{F}$ (force) gives us work)

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$\pm 1C = 6.24 \times 10^{18}$ protons or e^-'s

Ex:

Given the force field $\bar{F} = 4\hat{a}_y$ (N). This force field moves a 1C of charge (protons or e^-) along the y-axis from $y=0$ to $y=2m$. in a straight path.
Find the work done by this field?

Ans:

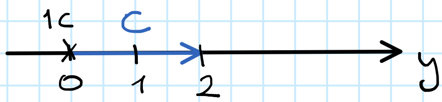
Solution Steps:

1-) Determine the proper coord. system : Rectangular.

Geometry = Curve. C.

2-) Use the formulas to find the unknown in the problem.

$\longrightarrow \bar{F} = 4\hat{a}_y$ (N)



$$W = \int_a^b \bar{F} \cdot d\bar{l}$$

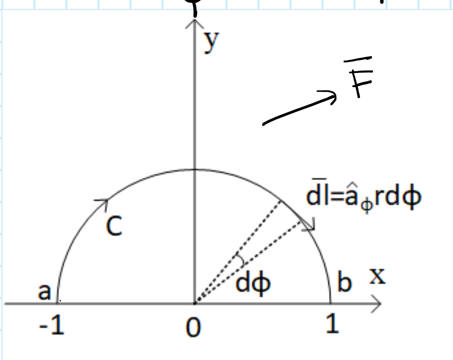
where $d\bar{l} = \hat{a}_x dx + \hat{a}_y dy + \hat{a}_z dz$

$$\Rightarrow \text{Work} = \int_a^b (4\hat{a}_y) \cdot (\hat{a}_x dx + \hat{a}_y dy + \hat{a}_z dz)$$

$$= \int_0^2 4\,dy = 4\int_0^2 dy = 4\,y\Big|_0^2 = 4\cdot(2-0) = 4\cdot 2 = 8 \text{ Joules.}$$

Ex:

Given $\bar{F} = \hat{a}_\varphi(-r\sin\varphi\cos\varphi + r\cos^2\varphi)$, the contour C is given as:



Find the work done in moving a 1C of charge from point a to point b along the contour C.

Ans:

1-) Coord. system? Geometry of the curve.

Cylindrical coord. is a proper choice for this problem.

2-) $W = \int_a^b \bar{F} \cdot d\bar{l}$ where $d\bar{l} = \cancel{\hat{a}_r dr} + \hat{a}_\phi r\,d\phi + \cancel{\hat{a}_z dz}$ (cylind. coord.)

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Now, we can substitute $\bar{F}$ and $\bar{dl}$ into the formula:

$$\text{Work} = \int_a^b \bar{F} \cdot \bar{dl} = \int_a^b \hat{a}_\varphi (-r \sin\varphi \cos\varphi + r \cos^2\varphi) \cdot \hat{a}_\phi r \, d\phi$$

$$= \int_{\phi=\pi}^{\phi=0} \left(-r^2 \sin\phi \cos\phi + r^2 \cos^2\phi\right) d\phi$$

where on C, r = 1 (half of the unit circle)

Thus,

$$W = \int_\pi^0 \left(\cos^2\varphi - \sin\varphi \cos\phi\right) \cdot d\phi = \int_0^{\pi} \left(\sin\varphi \cos\varphi - \cos^2\varphi\right) d\phi$$

$$= -\frac{\pi}{2} \text{ Joules.}$$

The minus sign refers to the work done against the field $\bar{F}$.

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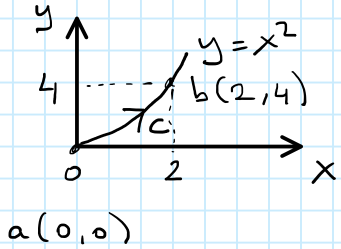
Ex:

Given $\bar{F} = \hat{a}_x x + \hat{a}_y xy$ as a force field.

and the curve is defined as:

C:



Find the work done by moving a 1C of charge from point a to point b.

Ans:

$$W = \int_a^b \bar{F} \cdot \bar{dl} \quad \text{where } \bar{dl} = \hat{a}_x dx + \hat{a}_y dy$$

$$W = \int_a^b (\hat{a}_x x + \hat{a}_y xy) \cdot (\hat{a}_x dx + \hat{a}_y dy)$$

$$W = \int_a^b x\,dx + xy\,dy = \int_a^b x\,dx + \int_a^b xy\,dy$$

The curve is given by $y = x^2$

$$\frac{dy}{dx} = 2x \Rightarrow dy = 2x\,dx$$

$$\Rightarrow W = \int_0^2 x\,dx + \int_0^2 x(x^2)\,2x\,dx$$

Then,

$$W = \frac{1}{2}x^2 \Big|_0^2 + 2\int_0^2 x^4\,dx = \frac{1}{2}(4) + 2\left(\frac{1}{5}x^5\right)\Big|_0^2$$

$$= 2 + \frac{2}{5}(2^5) = 2 + \frac{64}{5} = 14.8 \text{ J}$$

2-) Surface Integral of Vector Fields:

Let us consider a surface S in a 3D space as shown below:

P14

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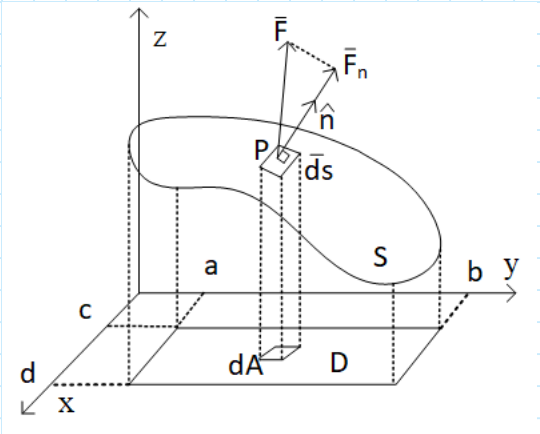


Suppose that there exists a vector field (force field) at every point in this space. Thus, $\bar{F}$ also exists at every point on S.

Consider a point P on S. We define $\bar{F}_n$ as the component of F normal to the surface at P.

We take a vector differential surface at point P as $\bar{ds}$. If we multiply $F_n \cdot \bar{ds}$ and sum this operation over S, we obtain the "surface integral of vector fields".

Thus,

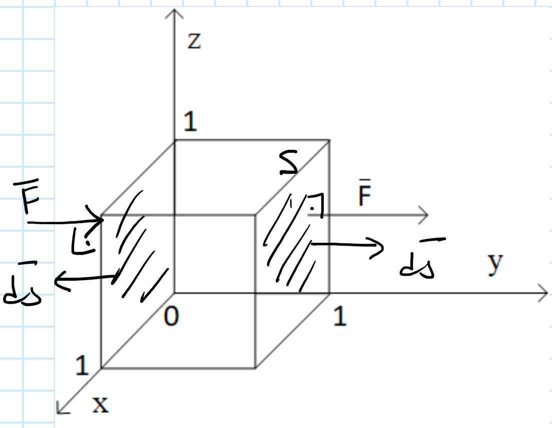
$$\text{Surface integral of vector field } \bar{F} = \iint_S \bar{F} \cdot \bar{ds}$$

We can also write this formula as:

$$\boxed{\text{Flux} = \int_S \bar{F} \cdot \bar{ds}}$$

Ex:

Given $\bar{F} = x^2\hat{a}_x + xy\hat{a}_y + yz\hat{a}_z$ (N), find the total flux that this force field creates on the surface of a unit cube located at the origin as shown below.



P15

9 Ekim 2020 Cuma 14:42

Ans:

1-) Use rectangular coord because of the cube.

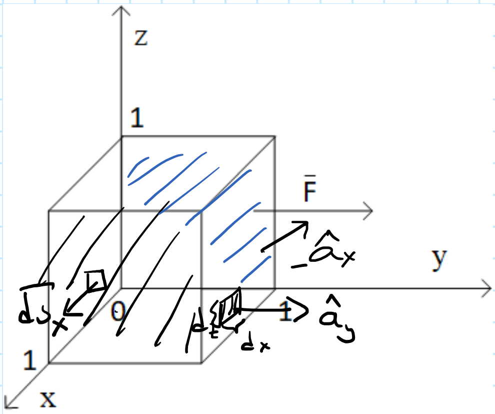
2-) $\text{Flux} = \int_S \bar{F} \cdot d\bar{s}$

$$\bar{F} = x^2\hat{a}_x + xy\hat{a}_y + yz\hat{a}_z$$

$S: S_1 + S_2 + \cdots + S_6$

For each surface, we will find the flux and at the end of the solution, we will sum the fluxes to find the total flux.

$\Rightarrow$ On the front face: $d\bar{s}_x = \hat{a}_x\, dy\, dz$, $x = 1$



$$\text{Flux}_1 = \iint_{S_1} \bar{F} \cdot d\bar{s}_x$$

$$= \iint_{S_1} (x^2\hat{a}_x) \cdot (\hat{a}_x\, dy\, dz)$$

$$\phi = \int_0^1 \int_0^1 x^2\, dy\, dz = 1$$

$x = 1$

For the back surface: $d\bar{s}_x = -\hat{a}_x\, dy\, dz$, $x = 0$,

$$\text{Flux}_2 = \int_0^1 \int_0^1 -x^2\, dy\, dz = 0 \;//$$

On the right surface: $d\bar{s}_y = \hat{a}_y\, dx\, dz$, $y = 1$,

$$\text{Flux}_3 = \int_0^1 \int_0^1 xy\, dx\, dz = \int_0^1 \int_0^1 x\, dx\, dz = \int_0^1 \left(\frac{1}{2}x^2 \Big|_0^1\right) dz = \frac{1}{2}\int_0^1 dz = \frac{1}{2} \;//$$

On the left surface: $d\bar{s}_y = -\hat{a}_y\, dx\, dz$, $y = 0$,

$$\text{Flux}_4 = \int_0^1 \int_0^1 -xy\, dx\, dz = 0.$$

On the top surface: $d\bar{s}_z = \hat{a}_z\, dx\, dy$, $z = 1$,

$$\text{Flux}_5 = \int_0^1 \int_0^1 yz\, dx\, dy = \frac{1}{2}.$$

On the bottom surface: $d\bar{s}_z = -\hat{a}_z\, dx\, dy$, $z = 0$

$$\text{Flux}_6 = \int_0^1 \int_0^1 -yz\, dx\, dy = 0.$$

$$\text{Flux}_{total} = \text{Flux}_1 + \text{Flux}_2 + \cdots + \text{Flux}_6 = 2 \;//$$

P1

20 Ekim 2022 Perşembe 20:06

Gradient of a Scalar Field:

∇ = Grad = Del

$$\boxed{\nabla f = \hat{a}_{u_1}\frac{\partial f}{h_1 \partial u_1} + \hat{a}_{u_2}\frac{\partial f}{h_2 \partial u_2} + \hat{a}_{u_3}\frac{\partial f}{h_3 \partial u_3}}$$

Koordinat Sistemi:	Kartezyen	Silindirik	Küresel
u_1	x	r	R
u_2	y	ϕ	θ
u_3	z	z	ϕ
h_1	1	1	1
h_2	1	r	R
h_3	1	1	R$sin\theta$

Table 1.1: Koordinat Sistemi Sabitleri

Ex:

Given a scalar function $f(x,y) = x^2 + y^2$ find the gradient of this function at point P(1,1) ?

$\nabla f(1,1) = ?$

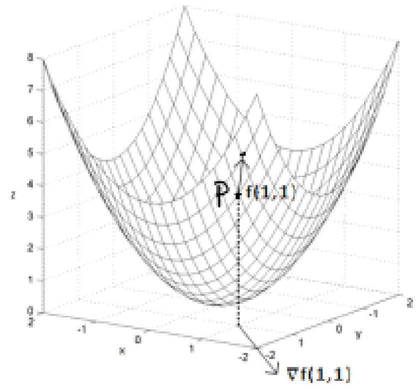
Ans:

$$\nabla f = \hat{a}_x \frac{\partial f}{\partial x} + \hat{a}_y \frac{\partial f}{\partial y} + \hat{a}_z \frac{\partial f}{\partial z}$$

$$= \hat{a}_x 2x + \hat{a}_y 2y$$

$$\nabla f \Big|_{P(1,1)} = \hat{a}_x 2 + \hat{a}_y 2$$

The graph of $f(x,y) = x^2 + y^2$ is given as :



- Thus, the gradient of f at P gives the vector that shows the max. change of f in that direction.

Divergence of a vector field:

$$\bar{F} = \hat{a}_{u_1} F_1 + \hat{a}_{u_2} F_2 + \hat{a}_{u_3} F_3$$

$$\boxed{\bar{\nabla} \cdot \bar{F} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} (h_2 h_3 F_1) + \frac{\partial}{\partial u_2} (h_1 h_3 F_2) + \frac{\partial}{\partial u_3} (h_1 h_2 F_3) \right]}$$

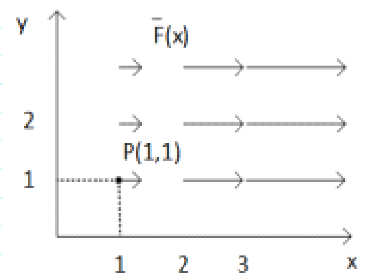
Ex:

Given the vector field $\bar{F}(x) = \hat{a}_x \underbrace{\frac{1}{2} x}_{F_x}$. Find $\bar{\nabla} \cdot \bar{F}$ at point $P(1,1)$?

Ans:

$$\bar{\nabla} \cdot \bar{F} \Big|_{P(1,1)} = \frac{\partial}{\partial x} F_x + \frac{\partial}{\partial y} \cancel{F_y}^{\,0} + \frac{\partial}{\partial z} \cancel{F_z}^{\,0} = \frac{\partial}{\partial x} \left(\frac{1}{2} x \right) \Big|_{P(1,1)} = \frac{1}{2} \cdot //$$

Let us draw the vector field $\bar{F} = \hat{a}_x \frac{1}{2} x$



- Therefore $\bar{\nabla} \cdot \bar{F} > 0$ means there is a source at P. and the value of $\bar{\nabla} \cdot \bar{F}$ gives us how much $\bar{F}$ diverges from the point P.

- $\bar{\nabla} \cdot \bar{F} \Big|_P = 0$, this means that there is $\overbrace{\text{no source}}^{\text{source free}}$ $\underbrace{\text{source}}_{\text{charge}}$ at point P.

Tuesday, October 25, 2022 9:03 AM

P3

20 Ekim 2022 Perşembe 20:36

Curl of a Vector Field:

Let $\bar{F} = \hat{a}_{u_1}F_1 + \hat{a}_{u_2}F_2 + \hat{a}_{u_3}F_3$ be $= \frac{1}{h_1h_2h_3}$ a vector field

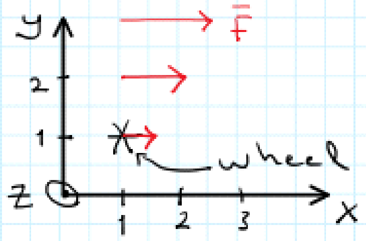
$$\boxed{\bar{\nabla}\times\bar{F} = \frac{1}{h_1h_2h_3}\begin{vmatrix} \hat{a}_{u_1}h_1 & \hat{a}_{u_2}h_2 & \hat{a}_{u_3}h_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1F_1 & h_2F_2 & h_3F_3 \end{vmatrix} = \hat{a}_{u_1}h_1\left\{\left[\frac{\partial}{\partial u_2}(h_3F_3) - \frac{\partial}{\partial u_3}(h_2F_2)\right] - \hat{a}_{u_2}h_2\left[\frac{\partial}{\partial u_1}(h_3F_3) - \frac{\partial}{\partial u_3}(h_1F_1)\right] + \hat{a}_{u_3}h_3\left[\frac{\partial}{\partial u_1}(h_2F_2) - \frac{\partial}{\partial u_2}(h_1F_1)\right]\right\}}$$

Ex:

Given the vector field $\bar{F}(x) = \hat{a}_x\frac{1}{2}y$, find $\bar{\nabla}\times\bar{F}$ at point $P(1,1)$. Explain its meaning?

Ans:

$$\bar{\nabla}\times\bar{F} = -\hat{a}_y\Big(\underbrace{\frac{\partial}{\partial x}(0)}_{0} - \underbrace{\frac{\partial}{\partial z}F_x}_{0}\Big) + \hat{a}_z\Big(\underbrace{\frac{\partial}{\partial x}(0)}_{0} - \frac{\partial}{\partial y}F_x\Big) = -\hat{a}_z\frac{\partial}{\partial y}\left(\frac{1}{2}y\right) = -\hat{a}_z\frac{1}{2}$$



$\bar{F} = \hat{a}_x\frac{1}{2}y$

The curl of a vector field gives the rate of rotation strength of the vector field at a point P.
If the rotation is clockwise, $\bar{\nabla}\times\bar{F} = -\hat{a}_z$ (into the screen).

P19

Wednesday, October 19, 2022 7:26 PM

Chapter 2 : Static Electric Field :

– Static means that the field (vector field = electric field) does <u>not</u> change with time. $\Rightarrow \frac{\partial}{\partial t} = 0$

– Electric field = Force acting on 1C of charge,
$\mp 1C = 1.654 \times 10^{15}$ $\underbrace{p^+ \text{ or } e^-}_{\text{charges.}}$

– Opposite charges attract, and the same charges repel each other. $\overleftarrow{F}+ \quad +\overrightarrow{F} \quad \overleftarrow{F}- \quad -\overrightarrow{F}$

$+\overrightarrow{F} \quad \overleftarrow{F}-$

– This force is called the "Coulumb's force".

– By, definition, the electric field is

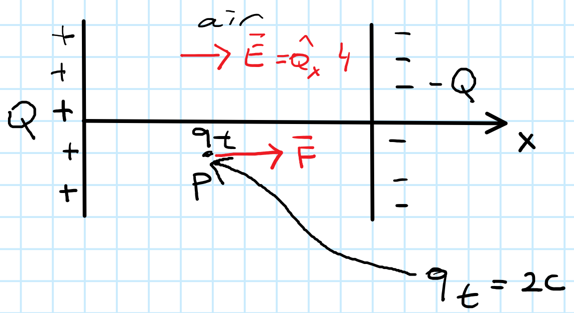
$$\boxed{\bar{E} = \frac{\bar{F}}{q_t} \quad \left(\frac{N}{C}\right)}$$

where the q_t is the test charge that experiences $\bar{F}$.

<u>Ex:</u>

A static electric field is created by placing two conducting plates facing one another, and filled by charges $+Q$ and $-Q$ respectively. This electric field is given as $\bar{E} = \hat{a}_x 4 \left(\frac{N}{C}\right)$. Find the force acting on a test charge $q_t = 2C$ that we bring into this field from outside?

<u>Ans:</u>



$$\bar{F} = \bar{E} \cdot q_t = \hat{a}_x 4 \left(\frac{N}{\cancel{C}}\right) \cdot (2\cancel{C}) = \hat{a}_x 8 (N)$$

P20

Monday, October 24, 2022 9:43 AM

Electric Potential (Voltage):

Voltage is the potential energy of a unit charge (1C) btw. two points in space.

1C = unit charge.

$q \quad \bar{E} = \frac{\bar{F}}{q}$

a b

Capacity Work .. Pot energy

$$V = \frac{\text{Work}}{q} = \frac{1}{q}\underbrace{\int_a^b \bar{F}\cdot\bar{dl}}_{\text{work}} = \int_a^b \frac{\bar{F}}{q}\cdot\bar{dl} = -\int_a^b \bar{E}\cdot\bar{dl}$$

This is the work per charge done by the force F.
The "−" sign is due to the fact that the work is done against the electric field. In other words, the unit charge has less energy at point b, and the energy increases towards point a.

$$\boxed{V = -\int_a^b \bar{E}\cdot\bar{dl}}$$

$$\boxed{U = V\cdot Q}$$

Charge. (Q)

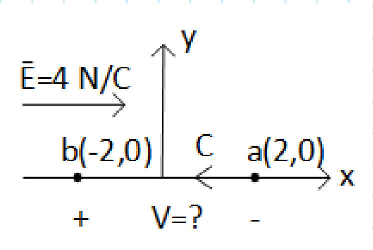
Electric Potential Energy (U)

Ex:

The electric field $\bar{E} = \hat{a}_x 4\ (\frac{N}{C})$ exists in a 2D space.
Find the electric potential (voltage) btw. points a(2,0) and b(−2,0).

Ans:

– The geometry of the problem is in rectangular coord.



$$V = -\int_a^b \bar{E}\cdot\bar{dl}$$

where $\bar{dl} = \hat{a}_x\, dx$

Then,

$$V = -\int^{-2} (\hat{a}_x 4)\cdot(\hat{a}_x\, dx)$$

or

$$V = -\int_2^{-2} 4\,dx = 4\int_{-2}^{2} dx = 4\left(x\Big|_{-2}^{2}\right) = 4\left[2-(-2)\right] = 4\cdot 4 = 16\ \left(\text{Volts} = \frac{J}{C}\right)$$

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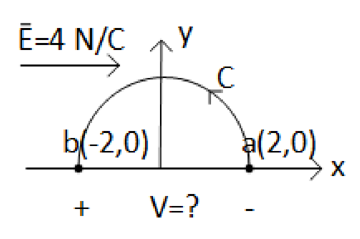
Monday, October 24, 2022 10:19 AM

Ex:

For the same electric field $\bar{E} = \hat{a}_x 4 \left(\frac{N}{C}\right)$, find the electric potential V btw. points b and a going through a contour C: half of a circle with radius 2, and its center is at the origin.

Ans:



The only difference in this problem is the path of the integration (the contour).

$$V = -\int_a^b \bar{E} \cdot \bar{dl}$$

- The proper coord. system in this problem is cylindrical. Thus, $\bar{dl} = \hat{a}_r \underset{0}{\cancel{dr}} + \hat{a}_\phi r\, d\phi + \underset{0}{\cancel{\hat{a}_z dz}}$. $= \hat{a}_\phi r\, d\phi$ for the given C.

- Electric field is given in rectangular coord. It needs to be converted into cylindrical coord as well.

We can use

$$E_r = E_x \cos\phi + \cancel{E_y}^{\,0} \sin\phi$$
$$E_\phi = -E_x \sin\phi + \cancel{E_y}^{\,0} \cos\phi$$
$$E_z = \cancel{E_z}^{\,0}$$

$\Rightarrow E_r = 4\cos\phi$, $E_\phi = -4\sin\phi$

$\Rightarrow \bar{E}(r,\phi,z) = \hat{a}_r 4\cos\phi - \hat{a}_\phi 4\sin\phi \cdot \left(\frac{N}{C}\right)$. (Electric field in cylindrical coord)

Then,

$$V = -\int_a^b \left(\hat{a}_r 4\cos\phi - \hat{a}_\phi 4\sin\phi\right) \cdot \left(\hat{a}_\phi r\, d\phi\right)$$

or

$$V = -\int_a^b \left(-4r\sin\phi\, d\phi\right) = \int_0^\pi 4\underset{\substack{\uparrow \\ 2}}{r} \sin\phi\, d\phi$$

P22

Monday, October 24, 2022 10:37 AM

or

$$V = 8\int_0^{\pi} \sin\phi \, d\phi = 8\left(-\cos\phi \Big|_0^{\pi}\right) = -8\big(\underbrace{\cos\pi}_{-1} - \underbrace{\cos 0}_{1}\big)$$

or

$$V = -8 \cdot (-2) = 16 \text{ V}.$$

- The electric potential is independent of the path.
- Thus, static electric field is conservative. (irrotational) ($\bar{\nabla} \times \bar{E} = 0$)

31.10.2022

- Take the gradient of both sides of the voltage equation

$$\bar{\nabla} V = -\bar{\nabla} \int_a^b \bar{E} \cdot \bar{dl}$$

Because of the fundamental theorem of calculus, differentiation and integration cancel.

$$\Rightarrow \boxed{\bar{E} = -\bar{\nabla} V} \quad \left(\frac{N}{C}\right).$$

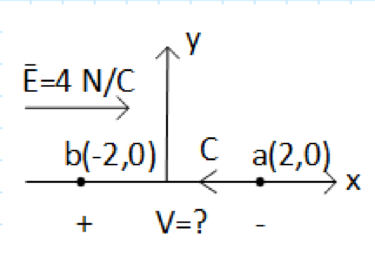
Ex:

Given $\bar{E} = \hat{a}_x 4 \left(\frac{N}{C}\right)$, find the voltage as a function of x between the points $a(2,0)$ and $b(-2,0)$. Given $V(a) = 0$ V.

Ans:

$V(x) = ?$ is asked ($V = V(b) - V(a) = 16$ V)

Using $\bar{E} = -\bar{\nabla} V$, we can find $V(x)$



- We use the rectangular coord.

$$\Rightarrow \hat{a}_x 4 = -\bar{\nabla} V = -\left[\hat{a}_x \frac{\partial}{\partial x} V(x)\right]$$

Thus,

$$\frac{d}{dx} V(x) = -4 \quad \text{(ordinary diff. equation.)}$$

Solving this equation:

$$dV(x) = -4\,dx$$

P23

Monday, October 31, 2022 9:44 AM

Take the integral of both sides:

$$\int d\,v(x) = -\int 4\,dx$$

or

$$v(x) = -4x + C$$, where C is the constant.

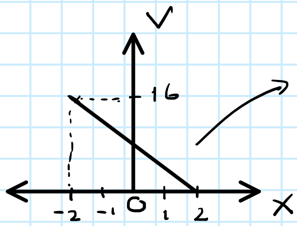
C is found using the initial condition $v(2) = 0$

Then, $0 = -4(2) + C$

$\Rightarrow C = 8$

Thus, $v(x) = -4x + 8$ (V)

Let us plot this voltage:



Linear voltage change.

—o—

Static Electric Field Calculations:

We have two important empirical formulas derived through experiments.

$$\boxed{\bar{\nabla}\cdot\bar{E} = \frac{\rho}{\epsilon_0}} \, , \quad \boxed{\bar{\nabla}\times\bar{E} = 0} \, .$$

ρ = free charge density $\left(\frac{C}{m^3}\right)$.

Also, $\epsilon_0 = 8.854\times10^{-12}$ is a constant.

Take the integral of both sides of these equations

For the 1st equation:

1-) Gauss' Law:

S: surface is closed $\longrightarrow$ $\boxed{\oint_S \bar{E}\cdot d\bar{s} = \frac{Q}{\epsilon_0}}$ (Gauss' Law.)

For the 2nd equation, we obtain

C: closed path $\longrightarrow$ $\boxed{\oint_C \bar{E}\cdot d\bar{l} = 0}$ (KVL)

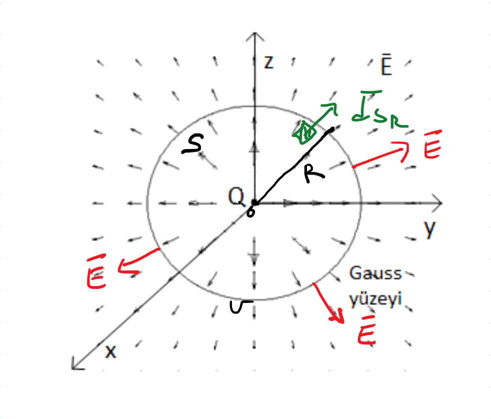
P24

10:45 AM

Ex:

Find the static electric field created by a point charge Q_o at a distance R away from the charge. in free space.

Ans:



- We place the charge Q_o at the origin.
- We can draw a sphere with the radius R and centered at the origin. The surface of this sphere is "Gaussian surface". Because, at every point on S, the electric field is the same

— By observation,

$$\bar{E} = \hat{a}_R \underbrace{E_R(R,\theta,\phi)}_{\text{unknown}} \quad \text{or} \quad \bar{E} = \hat{a}_R E_R$$

Let us use Gauss law

$$\oint_S \bar{E} \cdot \bar{ds} = \frac{Q}{\epsilon_o}$$

Total charge inside S.

Gauss surface → S

on S.

$$\int_0^{2\pi}\int_0^{\pi} \hat{a}_R E_R \cdot \hat{a}_R R^2 \sin\theta \, d\theta \, d\phi = \frac{Q_o}{\epsilon_o}$$

$$E_R R^2 \int_0^{2\pi}\int_0^{\pi} \sin\theta \, d\theta \, d\phi = \frac{Q_o}{\epsilon_o}$$

$$E_R R^2 4\pi = \frac{Q_o}{\epsilon_o}$$

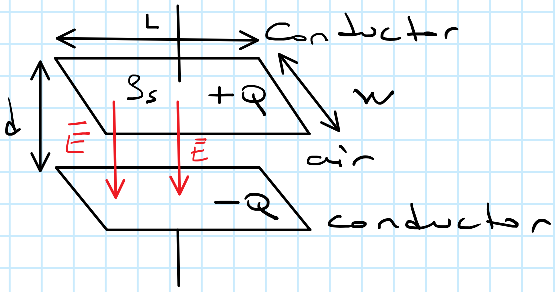
or $\boxed{E_R = \frac{Q_o}{4\pi\epsilon_o R^2}}$ $\left(\frac{N}{C}\right)$ Electric field created by a point charge Q_o.

P25

Monday, November 7, 2022 9:23 AM

Ex:

For the given geometry, find the static electric field inside the region between the conductors.



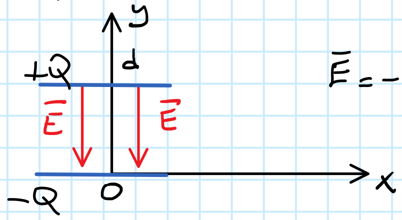
Given that $\rho_s\left(\frac{C}{m^2}\right)$ is the surface charge density.

$\Rightarrow$ There are +Q and −Q charges on each conductor plate.

Ans:

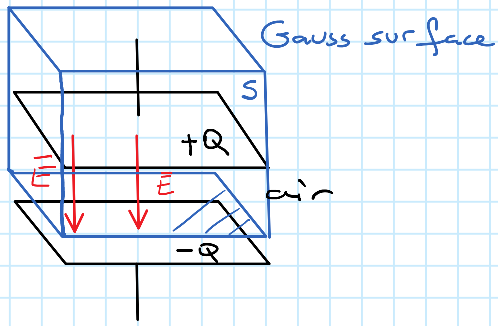
Step 1: Use rectangular coord.

Step 2: The problem in 2D is



$\bar{E} = -\hat{a}_y E_y$ from the geometry.

Step 3: Can we use "Gauss law"? Yes.



$$\oint_S \bar{E} \cdot d\bar{s} = \frac{Q}{\epsilon_0}$$

$$= \underbrace{\int_{S_1} \bar{E}_1 \cdot d\bar{s}_1}_{\text{bottom surface}} + \cancel{\int_{S_2} \bar{E}_2 \cdot d\bar{s}_2}^{\,0} + \cdots + \cancel{\int_{S_6} \bar{E}_6 \cdot d\bar{s}_6}^{\,0}$$

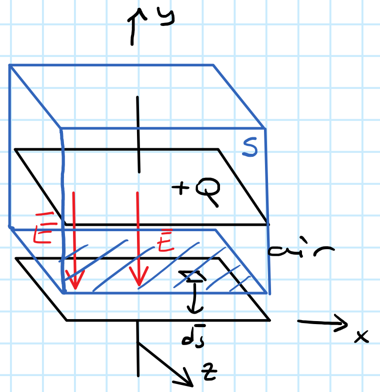
- On the top surface, $E = 0$.
- On the side surfaces, E is tangential to the surface.

  $\Rightarrow \bar{E} \cdot d\bar{s} = 0$, for side surfaces.
- The only surface integral, which is non-zero is the bottom surface.

P26

Monday, November 7, 2022 9:41 AM

For the bottom surface,



$$\oint_{S=S_1+S_2+\dots+S_6} \bar{E}\cdot\bar{ds} = \int_{S_{bottom}} \bar{E}\cdot\bar{ds} = \frac{Q}{\epsilon_0}$$

$\bar{ds} = -\hat{a}_y\, dx\, dz$, Q = Total charge inside the Gauss volume $= \rho_s\,(WL)$ (c).

$\bar{E} = -\hat{a}_y E_y$ $\qquad WL$: plate area.

Thus,

$$\int_{-W/2}^{W/2}\int_{-L/2}^{L/2} (-\hat{a}_y E_y)\cdot(-\hat{a}_y\, dx\, dz) = \frac{+Q}{\epsilon_0}$$

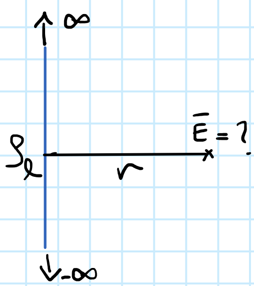
$$\Rightarrow E_y \underbrace{\int_{-W/2}^{W/2}\int_{-L/2}^{L/2} dx\,dz}_{\cancel{WL}} = \frac{\rho_s \cancel{WL}}{\epsilon_0} \Rightarrow \boxed{E_y = \frac{\rho_s}{\epsilon_0}\ \left(\frac{V}{m}\right).}$$

In terms of the total charge $\boxed{E_y = \frac{\rho_s WL}{\epsilon_0 WL} = \frac{Q}{\epsilon_0 S}\ \left(\frac{V}{m}\right)}$

Ex:

Determine the E-field of a very long straight line charge of uniform ρ_ℓ in air (free space).

Ans:



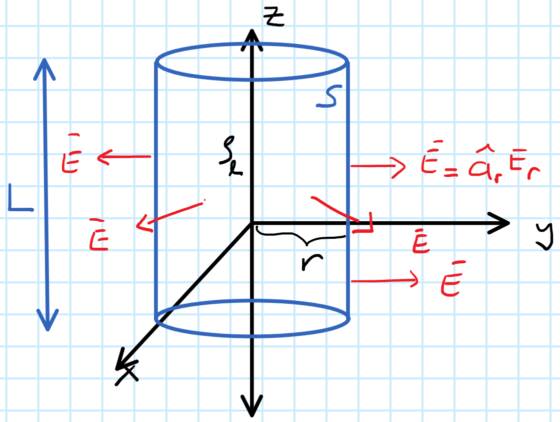
$\rho_\ell\,(\frac{C}{m})$ = Line charge density.

Can we apply Gauss law to find $\bar{E}$?

– Yes, we can.

P27

Monday, November 7, 2022 9:57 AM



1-) Cylindrical coord. system is proper to use.

2-) Define a Gauss surface. Gauss surface is the blue cylinder. On the side surface of the cylinder
$\bar{E} = \hat{a}_r E_r$ is uniform.
$\Rightarrow$ This enables us to use Gauss law

3-) Find the E-field

$$\oint_S \bar{E} \cdot d\bar{s} = \frac{Q}{\epsilon_0}$$

$\hookrightarrow$ Surface of the cylinder

On the top and bottom surfaces, $\bar{E} \cdot d\bar{s} = 0$.

Top surface:

$d\bar{s} = \hat{a}_z r\, dr\, d\phi$

$\hookrightarrow \bar{E} = \hat{a}_r E_r$ $\Rightarrow \bar{E} \cdot d\bar{s} = (\hat{a}_r E_r) \cdot (\hat{a}_z r\, dr\, d\phi) = 0$

— The only non zero integral comes from the side surface.

Thus,

$$\int_{S_{side}} \bar{E} \cdot d\bar{s} = \frac{Q}{\epsilon_0} \Rightarrow \int_{-L/2}^{L/2} \int_0^{2\pi} (\hat{a}_r E_r) \cdot (\hat{a}_r r\, d\phi\, dz)$$

Then,

$$E_r r \int_{-L/2}^{L/2} \int_0^{2\pi} d\phi\, dz = \frac{(\rho_\ell \cdot L)}{\epsilon_0}, \quad Q = \rho_\ell L \,(C)$$

$$E_r r \cdot (\cancel{L} \cdot 2\pi) = \frac{\rho_\ell \cdot \cancel{L}}{\epsilon_0}$$

$$\boxed{\Rightarrow E_r = \frac{\rho_\ell}{2\pi \epsilon_0 r} \quad \left(\frac{V}{m}\right)}$$

P28

Monday, November 7, 2022 9:59 AM

In case we can not apply the Gauss' law, the "integration technique" can be used. This is more general method to find the electric field from a given charge.

2-1 Integral Technique:

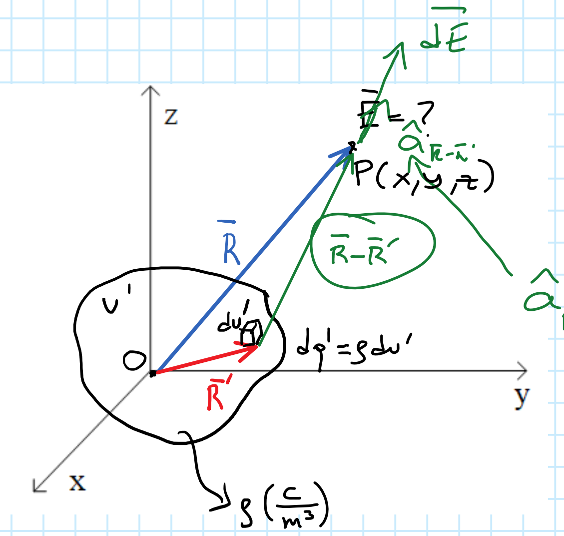
In this technique, we use one of the following equations to find the $\bar{E}$ at the point P.

$$\boxed{\bar{E} = \frac{1}{4\pi\epsilon_0} \int_{v'} (\bar{R}-\bar{R}') \cdot \frac{\rho}{|\bar{R}-\bar{R}'|^3} dv'}$$

$$\boxed{\bar{E} = \frac{1}{4\pi\epsilon_0} \int_{s'} (\bar{R}-\bar{R}') \cdot \frac{\rho_s}{|\bar{R}-\bar{R}'|^3} ds'}$$

$$\boxed{\bar{E} = \frac{1}{4\pi\epsilon_0} \int_{c'} (\bar{R}-\bar{R}') \cdot \frac{\rho_l}{|\bar{R}-\bar{R}'|^3} dl'}$$

V': Charge volume (m^3)
The prime is used for source variables.



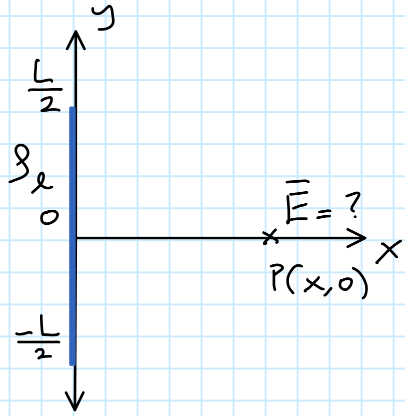
$$\hat{a}_{\bar{R}-\bar{R}'} = \frac{\bar{R}-\bar{R}'}{|\bar{R}-\bar{R}'|}$$

P29

Monday, November 7, 2022 10:46 AM

Ex:

Determine the electric field of a straight line of a conductor of length L. The conductor has a charge density ρ_ℓ. Plot the $|\bar{E}|$ by Matlab and verify the results by Ansys Electronics Desktop software.



$$\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{a}{\left(a^2+x^2\right)^{1.5}} dx = \frac{2L}{a\sqrt{L^2+4a^2}}$$

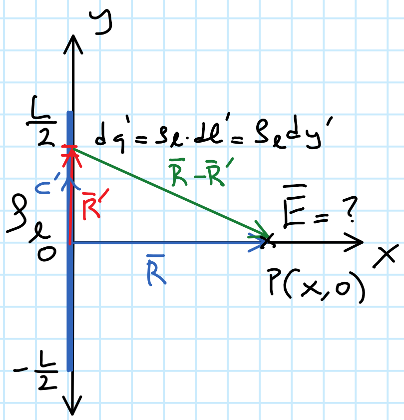
$$\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{x}{\left(a^2+x^2\right)^{1.5}} dx = 0$$

Ans:

We can not use the Gauss' law. Thus, we will use the integration technique.

Step 1: Use the rectangular coord.

Step 2: Define $\bar{R}$, $\bar{R}'$ and $\bar{R}-\bar{R}'$



$\bar{R} = \hat{a}_x x + \hat{a}_y \cancel{y}^{\,0} + \hat{a}_z \cancel{z}^{\,0} = \hat{a}_x x$

$\bar{R}' = \hat{a}_y y'$

$\bar{R}-\bar{R}' = \hat{a}_x x - \hat{a}_y y'$

$|\bar{R}-\bar{R}'| = \sqrt{x^2+(-y')^2}$

$= \sqrt{x^2+y'^2}$

Step 3: Use the integral formula

$$\bar{E} = \frac{1}{4\pi\epsilon_0}\int_{c'} (\bar{R}-\bar{R}')\cdot\frac{\rho_\ell}{|\bar{R}-\bar{R}'|^3} d\ell'$$

or

$$\bar{E} = \frac{\rho_\ell}{4\pi\epsilon_0}\int_{-L/2}^{L/2} \frac{(\hat{a}_x x - \hat{a}_y y')}{(x^2+y'^2)^{3/2}} dy'$$

or

$$\bar{E} = \frac{\rho_\ell}{4\pi\epsilon_0}\left[\hat{a}_x x \int_{-L/2}^{L/2} \frac{dy'}{(x^2+y'^2)^{3/2}} - \hat{a}_y \int_{-L/2}^{L/2} \frac{y'\,dy'}{(x^2+y'^2)^{3/2}}\right]$$

P30

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Using the integration table or www.symbolab.com, the integrals are evaluated as

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{a}{\left(a^2+x^2\right)^{1.5}}dx = \frac{2L}{a\sqrt{L^2+4a^2}} \quad \text{and} \quad \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{x}{\left(a^2+x^2\right)^{1.5}}dx = 0$$

($a=x$ and $x=y'$)

This result is also observable from the symmetry in the graph.

Thus,

$$\boxed{\bar{E} = \hat{a}_x \frac{\rho_\ell}{4\pi\epsilon_0}\frac{2L}{x\sqrt{L^2+4x^2}}}\left(\frac{N}{C}\right)=\left(\frac{V}{m}\right)$$

For example at $P(L,0)$

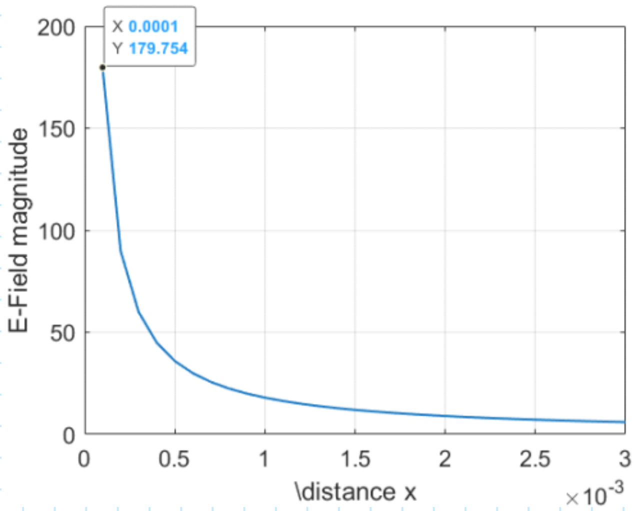
$$\bar{E}(L,0) = \hat{a}_x \frac{\rho_\ell}{\underset{2}{\cancel{4}}\pi\epsilon_0}\frac{\cancel{2}\cancel{L}}{\cancel{L}\sqrt{L^2+4L^2}} = \hat{a}_x \frac{\rho_\ell}{2\pi\epsilon_0}\frac{1}{L\sqrt{5}}\left(\frac{V}{m}\right)$$

Let us plot $|\bar{E}(x)| = \frac{\rho_\ell}{4\pi\epsilon_0}\frac{2L}{x\sqrt{L^2+4x^2}}$ w.r.t x:

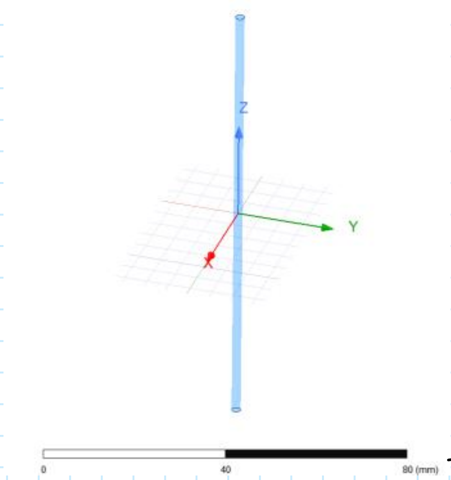
Matlab code:

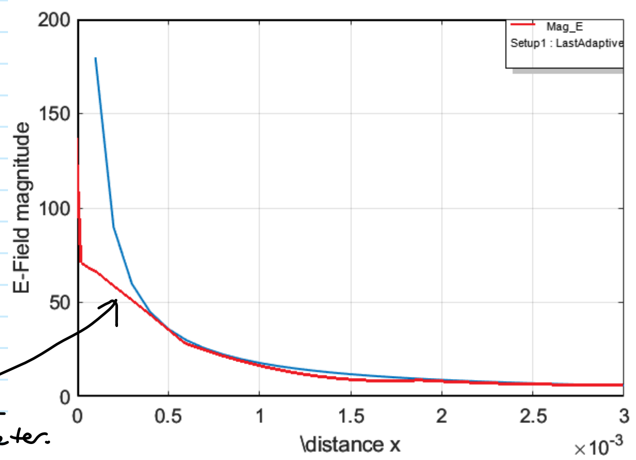
```
%% Electric Field question
clear all;
%% Parameters
rho_l=1e-12; % charge density
eps0=8.854*1e-12; % permittivity constant
x=0:0.1:3; % x variable
L=0.1; % 10 cm wire
E_field=(rho_l/(4*pi*eps0))*(2*L./(x.*sqrt(L.^2+4*x.^2)));
%% Plotting the E-field
plot(x,E_field,"LineWidth",1.5);
grid on;
set(gca,'FontSize', 14);
xlabel('\distance x');
ylabel('E-Field magnitude');
```

Result:



Ansys Results:





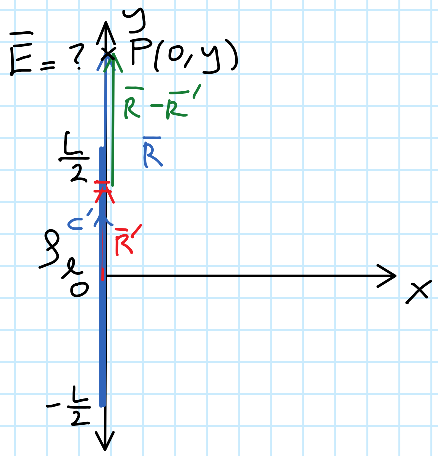
Sim. error is due to wire diameter.

P31

Monday, November 14, 2022 10:27 AM

Ex:

Find the static electric field at point P(0, y) for the charge density ρ_ℓ shown below.



Given that

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{y - x}{(y - x)^3} dx = \frac{4L}{(2y - L)(L + 2y)}$$

Ans:

$\bar{R} = \hat{a}_y y$

$\bar{R}' = \hat{a}_y y'$

$\bar{R} - \bar{R}' = \hat{a}_y (y - y')$

$|\bar{R} - \bar{R}'| = \sqrt{(y - y')^2}$

$= (y - y')$

$$\bar{E} = \frac{\rho_\ell}{4\pi\epsilon_0} \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{\hat{a}_y \cancel{(y - y')}}{(y - y')^{\cancel{3}\,2}} dy'$$

$$\bar{E}(0, y) = \frac{\rho_\ell}{4\pi\epsilon_0} \left[\hat{a}_y \left(\frac{4L}{(2y - L)(L + 2y)} \right) \right], \quad |\bar{E}| \propto \frac{1}{y}$$

P32

Monday, November 21, 2022 9:28 AM

The integration technique can also be applied using the voltage

$$\boxed{V\Big|_P = \frac{1}{4\pi\varepsilon_0}\int_{v'} \frac{\rho}{|\bar{R}-\bar{R}'|}\,dv'} \quad , \quad V = -\int_{\infty}^{P} \bar{E}\cdot\bar{dl}$$

$$\boxed{V = \frac{1}{4\pi\varepsilon_0}\int_{s'} \frac{\rho_s}{|\bar{R}-\bar{R}'|}\,ds'}$$

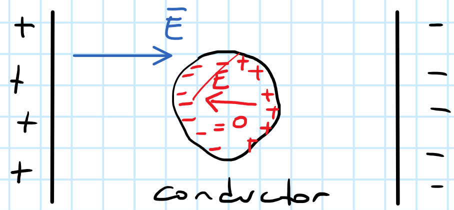
$$\boxed{V = \frac{1}{4\pi\varepsilon_0}\int_{c'} \frac{\rho_\ell}{|\bar{R}-\bar{R}'|}\,d\ell'}$$

Then, using $\boxed{\bar{E} = -\bar{\nabla} V}$ we can obtain the electric field.

- This voltage technique is used more often, because the integral is easier to evaluate.

—o—

Conductors in Static Electric Field :

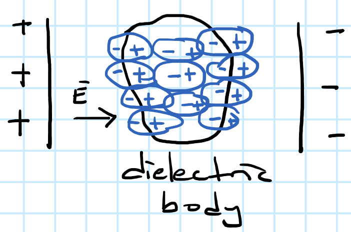


$\boxed{\bar{E} = 0}$ inside a conductor.

- Inside conductors, electric field is zero.

Dielectrics in Static Electric Field :

- Dielectrics do not contain free charges like conductors.
- Under an electric field, dielectric molecules polarize.



- Each polarized molecule can be considered as a "dipole moment". $\bar{p} = \bar{d}q$, $\bar{d}\downarrow^{-q}_{+q}$
- Define "Polarization vector" $= \bar{P}$ as

$$\bar{P} = \lim_{\Delta v \to 0} \frac{\sum_k \bar{p}_k}{\Delta v} = \text{Volume density of the dipole moments.}$$

P33

Monday, November 21, 2022 9:52 AM

We define a vector:

$\boxed{\bar{D} = \epsilon \bar{E}}$ as the "displacement vector".

where $\boxed{\epsilon = \epsilon_r \epsilon_0}$ is a constant, and is called the "permittivity".

ϵ is proportional to $\bar{P}$.

Then, we have

$$\boxed{\bar{\nabla} \cdot \bar{D} = \rho}$$

Then, the Gauss law becomes:

$$\boxed{\oint_S \bar{D} \cdot \bar{ds} = Q}$$ (Generalized Gauss' Law)

Ex:

Find the $\bar{E}$-field created by a unit charge Q_0 inside sea water.

$\epsilon_r = 82$ for sea-water.

Ans:

$$\int_0^{2\pi}\int_0^{\pi} \hat{a}_R D_R \cdot \hat{a}_R R^2 \sin\theta \, d\theta \, d\varphi = Q_0$$

$$D_R \cdot R^2 \underbrace{\int_0^{2\pi}\int_0^{\pi} \sin\theta \, d\theta \, d\varphi}_{4\pi} = Q_0$$

$$D_R 4\pi R^2 = Q_0$$

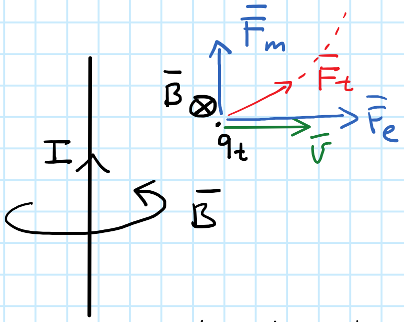
$$\Rightarrow D_R = \epsilon_r \epsilon_0 E_R = \frac{Q_0}{4\pi R^2} \qquad \Rightarrow E_R = \frac{Q_0}{4\pi \epsilon_0 \epsilon_r R^2} \left(\frac{V}{m}\right)$$

Relative Permittivity (ϵ_r) of some materials:

	ϵ_r	
Free space	1	($\bar{P} = 0 \rightarrow$ no polarization)
Air	1.00059	
Glass	4-10	
Paper	2-4	
Rubber	2.3-4	
Water (distilled)	80	(water polarizes the most)
Sea water	72	

– Static Magnetic Field –

Consider a straight wire with a constant current I



$$\bar{F}_m = q_t \cdot (\vec{v} \times \bar{B})$$

where q_t = test charge
$\vec{v}$ = velocity of the test charge
$\bar{B}$ = Magnetic field density vector created by current I.

– Unit for $\bar{B}$ is $\left(\frac{Wb}{m^2}\right)$, $1\,\frac{Wb}{m^2} = 1$ Tesla $= 10^4$ Gauss.

$$\Rightarrow \boxed{\bar{F}_{total} = \bar{F}_e + \bar{F}_m = q_t\bar{E} + q_t(\bar{v} \times \bar{B})}$$

or $\boxed{\bar{F}_t = q_t\left[\bar{E} + (\bar{v} \times \bar{B})\right]}$ (Lorentz force equation)

Two emperical formulas:

1-) $\bar{\nabla} \cdot \bar{B} = 0$
2-) $\bar{\nabla} \times \bar{B} = \mu_0 \bar{J}$
} point forms.

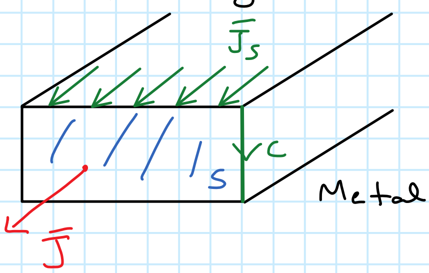
Integral forms:

1-) $\oint_s \bar{B} \cdot d\bar{s} = 0$

2-) $\boxed{\oint_c \bar{B} \cdot d\bar{l} = \mu_0 I \text{ (Ampere's law)}}$

Electric Current:

$I = \frac{Q}{t}$ (Ampere), Q = charge (C), t = time (sec.)



$\bar{J}$ = Current density vector. $\left(\frac{A}{m^2}\right)$, $\bar{J}_s$ = Surface current density $\left(\frac{A}{m}\right)$

where

$$\boxed{I = \int_s \bar{J} \cdot d\bar{s}} \qquad \boxed{I = \int_c \bar{J}_s \cdot d\bar{l}}$$

P35

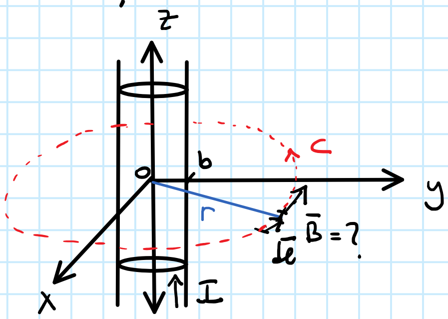
Monday, November 28, 2022 9:43 AM

Ex:

A long straight conductor with a circular cross section of radius b carries a steady current I, determine the magnetic flux density $\bar{B}$ both inside and outside the conductor.

Ans:

1-) Use cylindrical coord.



_ To be able to use Ampere's law, the closed contour must be selected such that $\bar{B}$ is the same at every point on the contour.

$$\bar{B}=\hat{a}_\phi B_\phi$$

Also, I is the current that penetrates through the area bounded by c.

For $r>b$:

$$\oint_c \bar{B}\cdot\bar{dl}=\mu_0 I$$

$$\bar{dl}=\hat{a}_\phi r\,d\phi$$

$$\int_0^{2\pi}(\hat{a}_\phi B_\phi)\cdot(\hat{a}_\phi r\,d\phi)=\mu_0 I$$

$$\int_0^{2\pi} B_\phi r\,d\phi=\mu_0 I \rightarrow B_\phi r\int_0^{2\pi} d\phi=\mu_0 I$$

$$\Rightarrow \boxed{B_\phi=\frac{\mu_0 I}{2\pi r}\ \left(\frac{Wb}{m^2}\right)}$$

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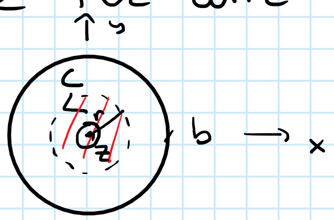
Monday, November 28, 2022 10:40 AM

For $r<b$:

Inside the wire

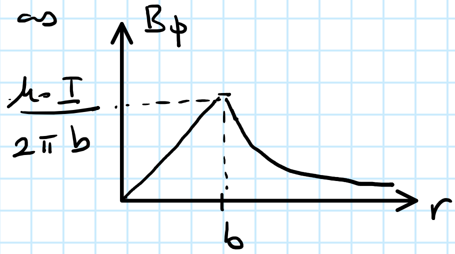


$$\int_0^{2\pi} (a_\phi B_\phi)\cdot(\hat{a}_\phi\, r\, d\phi) = \mu_o\left(\frac{\cancel{\pi} r^2}{\cancel{\pi} b^2}\right) I$$

$$\Rightarrow B_\phi\, r\,(2\pi) = \mu_o\left(\frac{r}{b}\right)^2 I$$

or $$B_\phi = \frac{\mu_o r I}{2\pi b^2}\left(\frac{w}{m^2}\right),\ r<b.$$

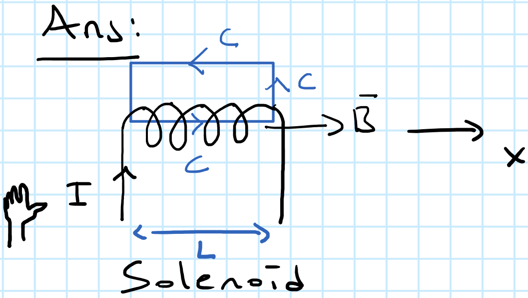
Thus, the total $\bar{B}$ can be shown as



Ex:

Determine the mag. field density vector $\bar{B}$ inside a solenoid with air core having n turns per unit length and carrying a current I.

Ans:



$\boxed{B_{outside} = 0}$ for a solenoid.

$$\oint_c \bar{B}\cdot\bar{dl} = \mu_o I \quad \text{(cartesian coord.)}$$

$$\int_{c_1} \bar{B}_1 \bar{dl}_1 + \int_{c_2} \bar{B}_2 \bar{dl}_2 + \cdots + \int_{c_4} \bar{B}_4 \bar{dl}_4$$

All integrals are zero except the contour on the x-axis.

$$\int_0^L (\hat{a}_x B_x)\cdot(\hat{a}_x\, dx) = \mu_o N I \quad , N = \text{total \# of turns.}$$

P37

Monday, November 28, 2022 11:09 AM

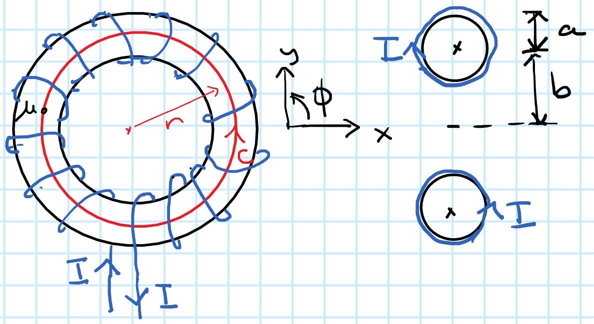
$B_x L = \mu_o N I$ or $\boxed{B_x = \frac{\mu_o N I}{L} \left(\frac{wb}{m^2}\right)}$, n = # of turns per unit length

$\boxed{B_x = \mu_o n \frac{\cancel{L} I}{\cancel{L}} = \mu_o n I \left(\frac{wb}{m^2}\right)}$, $n = \frac{N}{L}$

Ex:

Determine the magnetic flux density inside a closely wound toroidal coil with an air core having N turns and carrying a current I. The toroid has a mean radius b, and radius of each turn is a.

Ans:



Toroid

Step 1:

The contour C: circle with radius r ; $b-a<r<b+a$

Step 2:

Use cylindrical coord. because C is circular.

3-) Use Ampere's law to find $\bar{B}$.

$$\oint_C \bar{B} \cdot \bar{dl} = \mu_o I$$, I = current that passes through the area enclosed by C.

where $\bar{dl} = \hat{a}_\phi \, r \, d\phi$

and $\bar{B} = \hat{a}_\phi B_\phi$

$$\Rightarrow \int_0^{2\pi} \hat{a}_\phi B_\phi \cdot \hat{a}_\phi \, r \, d\phi = \mu_0 N I \quad , \; N = \text{total \# of turns.}$$

$$B_\phi \, r \, 2\pi = \mu_0 N I$$

$$\Rightarrow B_\phi = \frac{\mu_0 N I}{2\pi r} \quad , \quad b-a < r < b+a$$

Vector Magnetic Potential:

This is analogues to voltage in electric field.

By definition,

$$\boxed{\bar{B} = \bar{\nabla} \times \bar{A}}$$

where $\bar{A}$ = Vector magnetic potential.

Then, $\bar{A}$ is similar to V in electrostatics, except it is a vector.

– Integration Technique –

We use $\bar{A}$ to solve problems to find B. This is true for non-symmetrical problems where the application of Ampere's law is not possible.

$\bar{A}$ can be obtained from sources (current) as

Refer to pp 233 for details! $\longrightarrow$

$$\boxed{\bar{A} = \frac{\mu_0}{4\pi} \int_{v'} \frac{\bar{J}}{|\bar{R}-\bar{R}'|} dv'} \qquad \text{(volumetric current source)}$$

or

$$\boxed{\bar{A} = \frac{\mu_0}{4\pi} \int_{s'} \frac{\bar{J}_s}{|\bar{R}-\bar{R}'|} ds'} \qquad \text{(surface current source)}$$

or

$$\boxed{\bar{A} = \frac{\mu_0}{4\pi} \int_{c'} \frac{\bar{I} \, dl'}{|\bar{R}-\bar{R}'|}} \qquad \text{(linear current source)}$$

– Once, we find $\bar{A}$, $\Rightarrow \bar{B} = \bar{\nabla} \times \bar{A}$.

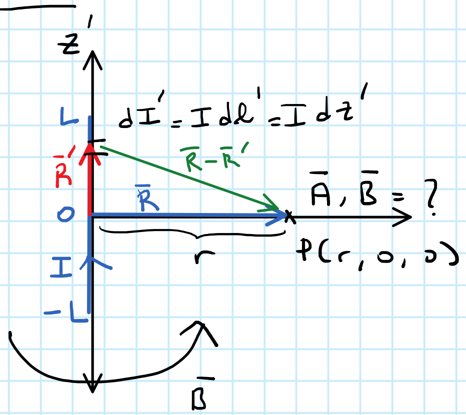
P39

Monday, December 5, 2022 9:56 AM

Ex:

A direct current I flows in a straight wire of length 2L. Find $\bar{B}$ at a point located at a distance r from the wire in the bi-secting plane.

Ans:



Then,

$$\bar{R} = r\hat{a}_r \quad \Rightarrow \bar{R}-\bar{R}' = r\hat{a}_r - z'\hat{a}_z \qquad \bar{A} = \frac{\mu_o}{4\pi}\int_C \frac{\bar{I}\,d\ell'}{|\bar{R}-\bar{R}'|}$$

$$\bar{R}' = z'\hat{a}_z \quad \Rightarrow |\bar{R}-\bar{R}'| = \sqrt{r^2+(z')^2}$$

Thus,

$$\bar{A} = \frac{\mu_o}{4\pi}\int_{-L}^{L}\frac{\bar{I}\,dz'}{[r^2+(z')^2]^{\frac{1}{2}}} = \frac{\mu_o \bar{I}}{4\pi}\underbrace{\int_{-L}^{L}\frac{dz'}{[r^2+(z')^2]^{\frac{1}{2}}}}_{\ln\left\{z'+[(z')^2+r^2]^{\frac{1}{2}}\right\}\Big|_{-L}^{L}}$$

This integral will be given in the question.

$$= \ln\frac{\sqrt{L^2+r^2}+L}{\sqrt{L^2+r^2}-L}.$$

$$\Rightarrow \bar{A} = \hat{a}_z\frac{\mu_o I}{4\pi}\ln\frac{\sqrt{L^2+r^2}+L}{\sqrt{L^2+r^2}-L}.$$

$$\Rightarrow \bar{B} = \bar{\nabla}\times\bar{A} = \bar{\nabla}\times(\hat{a}_z A_z) = \hat{a}_r\frac{1}{r}\underbrace{\cancel{\frac{\partial A_z}{\partial\phi}}}_{0} - \hat{a}_\phi\frac{\partial A_z}{\partial r}$$

$$= -\hat{a}_\phi\frac{\partial}{\partial r}\left[\frac{\mu_o I}{4\pi}\ln\frac{\sqrt{L^2+r^2}+L}{\sqrt{L^2+r^2}-L}\right]$$

$$\boxed{\bar{B} = \hat{a}_\phi\frac{\mu_o I L}{2\pi r\sqrt{L^2+r^2}}.}$$

Now, if $r \ll 2L$ (infinitely long wire assumption)

$$\Rightarrow L^2+r^2 \simeq L^2$$

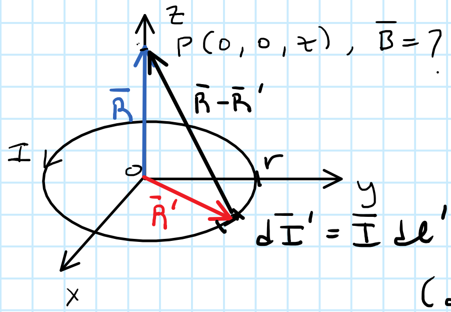
$$\Rightarrow \bar{B} = \hat{a}_\phi\frac{\mu_o I \cancel{L}}{2\pi r\cancel{L}} = \hat{a}_\phi\frac{\mu_o I}{2\pi r}\left(\frac{Wb}{m^2}\right) \text{ as before.}$$

Ex:

Find the magnetic flux density at a point on the axis of a circular loop of radius r that carries a direct current I.

Ans:

Using cylindrical coord.



$$d\bar{I}' = \bar{I}\, d\ell' = \hat{a}_\phi I r\, d\phi'$$
$$(d\ell' = r\, d\phi')$$

$$\Rightarrow \bar{R} = \hat{a}_z z$$
$$\bar{R}' = \hat{a}_r b$$
$$\bar{R} - \bar{R}' = \hat{a}_z z - \hat{a}_r r$$
$$|\bar{R} - \bar{R}'| = \sqrt{z^2 + r^2}$$

$$\Rightarrow \bar{A} = \frac{\mu_0}{4\pi} \int_0 \frac{\bar{I}\, d\ell'}{\sqrt{z^2 + r^2}} ,$$

$$\Rightarrow \bar{A} = \frac{\mu_0}{4\pi} \int_0^{2\pi} \frac{\hat{a}_\phi I}{[z^2 + r^2]^{\frac{1}{2}}}\, d\phi' = \hat{a}_\phi \frac{\mu_0 I}{4\pi \sqrt{z^2 + r^2}} \underbrace{\int_0^{2\pi} d\phi}_{2\pi}$$

$$= \hat{a}_\phi \underbrace{\frac{\mu_0 I r}{2\sqrt{z^2 + r^2}}}_{A_\phi}$$

and

$$\bar{B} = \bar{\nabla} \times \bar{A} = \frac{1}{r} \begin{vmatrix} \hat{a}_r & \hat{a}_\phi r & \hat{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & r A_\phi & 0 \end{vmatrix}$$

− $\frac{1}{r}$ factor is multiplied by each element of 3rd row, then the derivatives are taken.

$$= \frac{1}{r} \left[\underbrace{\hat{a}_r \left[-\frac{\partial}{\partial z}(r A_\phi) \right]}_{0 \text{ (due to symmetry.)}} + \hat{a}_z \left[\frac{\partial}{\partial r}(r A_\phi) \right] \right] = \frac{1}{\cancel{r}} \hat{a}_z \frac{\partial}{\partial r}(\cancel{r} A_\phi)$$

$$= \hat{a}_z \frac{\mu_0 I r^2}{2(z^2 + x^2)^{3/2}}$$

$$\bar{B} = \hat{a}_z \frac{\mu_0 I r^2}{2(z^2 + r^2)^{3/2}} \left(\frac{Wb}{m^2} \right)$$

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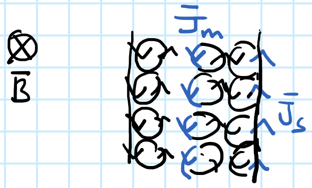
Magnetization:

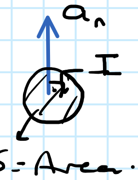
Consider the material,



When inserted into a mag. field density vector $\bar{B}$, some materials (magnetic materials) show electron spin behavior.

Thereby creating surface current density $\bar{J}_s$ and current density $\bar{J}_m$ inside the material.

Define $\bar{m}$ = Magnetic dipole moment as



$\Rightarrow \quad \bar{m} = \hat{a}_n I S$

We also define,

$$\bar{M} = \frac{\Sigma \bar{m}}{dv} \quad \text{(magnetization vector)}$$

We define

$\bar{B} = \mu_0 \mu_r \bar{H} = \mu \bar{H}$, where μ: Permeability

and

$$\boxed{\bar{H} = \frac{\bar{B}}{\mu}} \quad , \mu = \mu_0 \mu_r \quad , \mu \propto \bar{M} \quad , \bar{H} = \text{Magnetic intensity vector.}$$

$$\boxed{\oint_c \bar{H} \cdot d\bar{l} = I} \quad \text{(General Ampere's law)}$$

For $\mu_r < 1 \longrightarrow$ Diamagnetic materials.

$\mu_r \gtrsim 1 \longrightarrow$ Paramagnetic materials.

$\mu_r >> 1 \longrightarrow$ Ferromagnetic materials. (Ferrites.)

↳ Ex: Fe, Ni, Zn, Co.

Fe is used mostly in applications.

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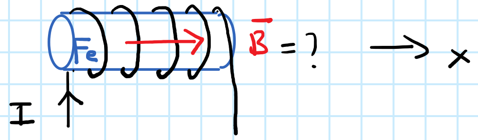
Monday, December 5, 2022 11:00 AM

Ex:

Solve the solenoidal example using an Fe core inside the structure (L=Length of the solenoid) ($\mu_r = 7000$)



Ans:

Using the general Ampere's law:

$$\oint_C \bar{H} \cdot \bar{dl} = NI$$

or

$$\int_0^L H_x \cdot dx = NI \quad \text{or} \quad H_x L = NI \quad \text{where } H_x = \frac{B_x}{\mu}$$

Then,

$$\frac{B_x}{\mu} \cdot L = NI \Rightarrow B_x = \frac{\mu N I}{L} = \mu n I$$

, n = # of turns per unit length.

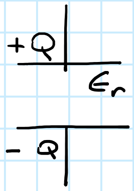
$$\Rightarrow B_x = \mu n I = \mu_0 \mu_r n I = 7000 \mu_0 n I \left(\frac{Wb}{m^2}\right)$$

$\Rightarrow$ As a result, the magnetic field inside the solenoid is increased by 7000 times due to the insertion of Fe. This is contrary to electric field, where we had a decrease in electric field inside dielectrics.

Capacitance & Capacitors:

A capacitor is a device with two conductors and a dielectric in btw. them.

Capacitance is the capacity of a capacitor to hold charges per voltage applied across its plates.



$$\boxed{C = \text{Capacitance} = \frac{Q}{V} \text{ (Farads)}}$$

- If $C = 2F$ means that for 1V of voltage applied across the capacitor, 2C of charges are collected at each plate.

Ex:

Determine the capacitance of a parallel plate capacitor with plate surface S and separation of d, filled by ϵ_r.

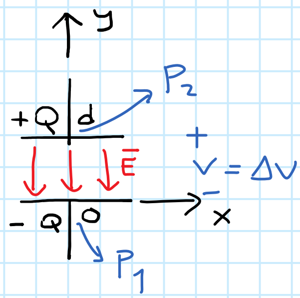
Ans: Steps to find Capacitance:

1-) Rectangular coord.

2-) Assume a charge Q and $-Q$ on the conductors.

Finding the E-field inside the dielectric.

$$E_y = \frac{Q}{\epsilon S}$$



3-) Find V:

$$V = -\int_{P_1}^{P_2} \bar{E} \cdot \bar{dl} = -\int_0^d (-\hat{a}_y E_y) \cdot (\hat{a}_y dy)$$

$$= \int_0^d E_y \, dy$$

$$= \int_0^d \frac{Q}{\epsilon S} dy = \frac{Q}{\epsilon S} d \text{ (V)}$$

4-) To find C, use:

$$C = \frac{Q}{V} = \frac{\cancel{Q}}{\cancel{Q}\frac{d}{\epsilon S}} = \frac{\epsilon S}{d} \Rightarrow C = \frac{\epsilon S}{d} \text{ (F)}.$$

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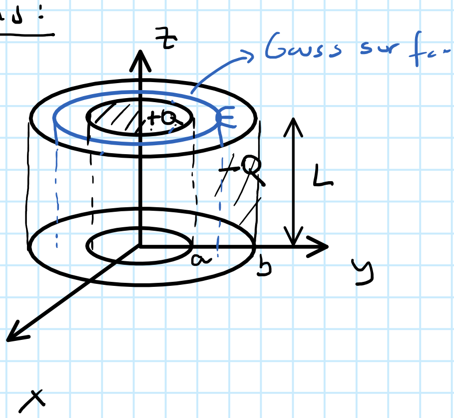
Ex:

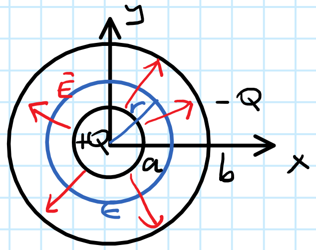
A cylindrical capacitor consists of an inner conductor of radius a, and an outer conductor of radius b. The space btw. the conductors is filled with ϵ. The length of the capacitor is L. Determine the capacitance?

Ans:



1-) Choose cylindrical coord.

2-) Assume +Q, -Q on the conductors. Find the E-field in btw. the conductors.



$\bar{E} = \hat{a}_r E_r$

Use Gauss law:

- On this surface Electric field is uniform. ✓

$$\oint_S \bar{D} \cdot \bar{ds} = Q$$

- On the top and bottom surfaces, $\bar{D} \cdot \bar{ds} = 0$. Only side surface contributes to the integral:

$$\iint_{\substack{\text{side} \\ \text{surface}}} (\hat{a}_r \epsilon E_r) \cdot (\hat{a}_r r \, d\phi \, dz) = Q$$

$$\Rightarrow \int_0^L \int_0^{2\pi} \epsilon E_r r \, d\phi \, dz = Q$$

$$\epsilon E_r r \int_0^L \int_0^{2\pi} d\phi \, dz = Q$$

$$\Rightarrow E_r = \frac{Q}{2\pi \epsilon L r} \quad \left(\frac{V}{m}\right)$$

Step 3: Find V:

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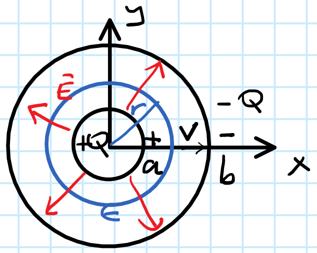
$$V_{ab} = -\int_{r=b}^{r=a} \bar{E}\cdot \bar{dl}$$



$$= -\int_b^a (\hat{a}_r E_r)\cdot(\hat{a}_r\, dr)$$

$$= -\int_b^a \frac{Q}{2\pi\epsilon L r}\, dr = \frac{Q}{2\pi\epsilon L}\int_a^b \frac{1}{r}\, dr = \frac{Q}{2\pi\epsilon L}\ln\left(\frac{b}{a}\right)$$

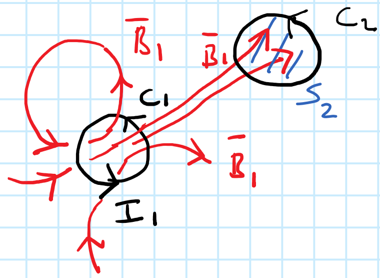
Step 4:
$$C = \frac{Q}{V} = \frac{\cancel{Q}}{\frac{\cancel{Q}}{2\pi\epsilon L}\ln\left(\frac{b}{a}\right)} = \frac{2\pi\epsilon L}{\ln\left(\frac{b}{a}\right)} \quad (F)$$

$\hookrightarrow$ Farads

Inductance & Inductors:

Suppose we have the following closed circuit with current I_1

$\bar{B}_1$ is the magnetic field density vector created by I_1. Now, if we bring another circuit into the field B_1



Some of $\bar{B}_1$ passes through the area enclosed by C_2.
How much of $\bar{B}_1$ passes through the area by C_2 is:

Φ_{12} = The flux on the 2nd circuit created by the 1st circuit.

$$\Phi_{12} = \int_{S_2} \bar{B}_1 \cdot \bar{dS}_2 \quad (Wb)$$

Since $B_1 \propto I_1 \Rightarrow \Phi_{12} \propto I_1$

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This proportionality constant is called the "inductance".

Then, $\boxed{L_{12} = \frac{\phi_{12}}{I_1}}$ (Mutual inductance)

In case C_2 has N turns:

$\boxed{L_{12} = \frac{N\phi_{12}}{I_1} = \frac{\Lambda_{12}}{I_1}}$, where Λ_{12} = Flux for N turns.

Self inductance:

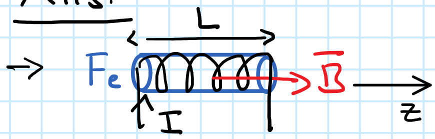
Flux linkage per unit current I_1 in the loop C_1 itself.

$$\Rightarrow \boxed{L_{11} = \frac{\Lambda_{11}}{I_1} \text{ (Henry)}}$$

Ex:

Find the inductance of a solenoid filled with iron core having n turns per unit length ($\mu_r = 7000$)

Ans:



Choke

$L = \frac{\Lambda}{I}$, where $\Lambda = nL\phi$

where $\phi = \int_S \bar{B} \cdot d\bar{s}$

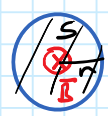
Step 1: Cylindrical coord. system.

Step 2: Find the magnetic field inside the structure.

$$\bar{B} = \hat{a}_z \mu n I = \hat{a}_z \mu \frac{N}{L} I \left(\frac{Wb}{m^2}\right)$$

Step 3: Find the flux ϕ

$$\Rightarrow \phi = \int_S \bar{B} \cdot d\bar{s} = B.S = S\mu n I \text{ (wb)}$$



$S = \pi r^2 (m^2)$

Step 4:

$$\Rightarrow \Lambda = N\phi = S\mu_0\mu_r \frac{N^2}{L} I \text{ (Wb)}$$

Step 5: Find the inductance L:

$$L = \frac{\Lambda}{I} = S\mu_0\mu_r \frac{N^2}{L} \, (H) \hookrightarrow \text{Henry}$$

As a result, the inductance is proportional to the number of turns.

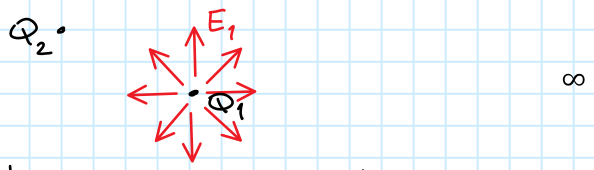
– Electrostatic Energy and Forces –

Suppose we have a number of charges so far apart that they can not interact with one another. We grab one charge and put it in our system:

$$\bullet \, Q_1$$

Bringing this charge takes no work since this is the only charge in our system. $\Rightarrow W_1 = 0$

When we pull a second charge Q_2 into the system:



Q_2 experiences a force due to E_1 around Q_1.

The work required to bring Q_2 from infinity to the system is

$$W_2 = Q_2 \cdot V_2$$

Then, $W_{tot} = W_1 + W_2 = 0 + Q_2V_2$ ——— (1)

Do the same process reversed, bring Q_2 first, then Q_1

$$\Rightarrow W_2 = 0 \,, \quad W_1 = Q_1 \cdot V_1$$

Then $W_{tot} = Q_1V_1 + 0$ ——— (2)

Sum (1) and (2)

$$\begin{array}{rl} & W_{tot} = 0 + Q_2V_2 \\ + & W_{tot} = Q_1V_1 + 0 \\ \hline & 2W_{tot} = Q_1V_1 + Q_2V_2 \end{array}$$

or $W_{tot} = \frac{1}{2}(Q_1V_1 + Q_2V_2)$

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If we have N charges in the system

$$\boxed{W_E = \frac{1}{2}\sum_{k=1}^{N} Q_k V_k}$$

(Total energy stored in an electrostatic system of N charges)

- Electrostatic Energy in terms of Field Quantities -

For a continuous charge distribution

$$Q\Big|_{dv} = \rho\, dv \quad , \rho = \text{volume charge density } \left(\frac{C}{m^3}\right)$$

$$W_E = \lim_{\Delta v \to 0} \frac{1}{2}\sum_{k=1}^{N} \rho \Delta v_k V_k = \frac{1}{2}\int_{v'} \rho V \, dv$$

(Electrostatic energy stored inside a volume v' by infinite charges.)

Since $\bar{\nabla}\cdot\bar{D} = \rho$

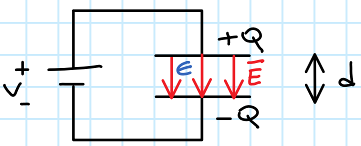
$$W_E = \frac{1}{2}\int_{v'} (\bar{\nabla}\cdot\bar{D})\, V\, dv$$

After some mathematical manipulations (p.137) textbook

$$\boxed{W_E = \frac{1}{2}\int_{v'} \epsilon E^2 \, dv}$$

where $w_e = \epsilon E^2$ = Energy density. ⇒ The energy is inside the E-field.

Ex:



Find the electrostatic energy stored inside the capacitor.

Ans:

$$E = \frac{V}{d}$$

Then,

$$W_E = \frac{1}{2}\int_{v'} \epsilon\left(\frac{V}{d}\right)^2 dv = \frac{1}{2}\epsilon\left(\frac{V}{d}\right)^2 (Sd) \quad , \text{where } S = \text{Plate surface area.}$$

$$= \frac{1}{2}\epsilon\frac{S}{d}V^2 = \boxed{\frac{1}{2}CV^2}\ (J) = \frac{1}{2}\left(\frac{Q}{V}\right)V^2 = \boxed{\frac{1}{2}QV}\ (J)$$

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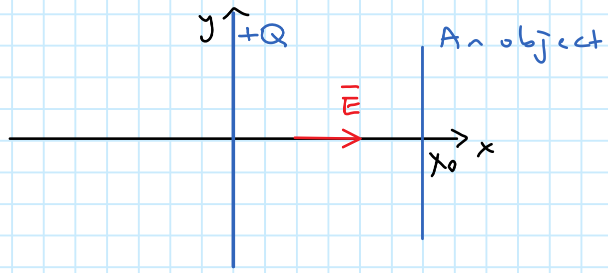
- Electrostatic Forces -

Suppose we have a static $\bar{E}$-field created by a very long straight conductive wire of static charge Q.

Along the x-axis the $\bar{E}$ field is $\bar{E} = \hat{a}_x E_x$

If we place an object at $x = x_0$, can we move it by the $\bar{E}$ field?



It turns out that the answer is yes! The $\bar{E}$-field has a momentum.

The momentum is defined as:

$$p = mv$$

where m = mass, v = velocity.

The $\bar{E}$-field is not a particle, thus it has no mass; however, it has an effective mass which can be found from

$$E = mc^2 \quad \text{(Einstein's formula)}$$

where m = mass, c = velocity

$$\Rightarrow m = \frac{E}{v^2}, \quad v = \text{velocity}.$$

Then,

$$\boxed{p = \frac{E}{v^{\cancel{2}}} \cdot \cancel{v} = \frac{E}{v}} \text{———} (1)$$

Also,

$$F = ma = m\frac{dv}{dt}$$

$$\Rightarrow F\,dt = m\,dv = dp \quad \text{(instantaneous momentum or differential momentum.)}$$

$$\text{Then, } \boxed{F = \frac{dp}{dt}} \quad \text{(Force-momentum relation)}$$

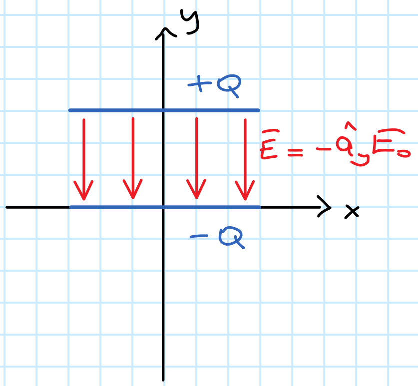
where $dp = \frac{dE}{dv}$ from (1),

$$\Rightarrow F = \frac{dE}{dv} \cdot \frac{1}{dt} \quad \text{or} \quad F\,\cancel{dt} = dE \cdot (dv)^{-1} = dE\left(\frac{\cancel{dt}}{dx}\right)$$

$$\Rightarrow \boxed{\bar{F} = \frac{dE}{dx} = -\frac{d\,W_E}{dx} = -\bar{\nabla} W_E}$$

Ex:

Find the force exerted on $-Q$ loaded plate due to the $\bar{E}$-field.



Ans:

Using the rectangular coordinates,

$$\bar{F} = -\bar{\nabla} W_E = -\left(\hat{a}_x \frac{d}{dx} W_E + \hat{a}_y \frac{d}{dy} W_E + \hat{a}_z \frac{d}{dz} W_E\right)$$

where $W_E = \frac{1}{2} QV$

Since Q is constant, we need to find the expression for V just like we did on pg. 22.

Thus, from $\bar{E} = -\bar{\nabla} V$, we can find $V(y)$

$$-\hat{a}_y E_o = -\bar{\nabla} V = -\left[\hat{a}_y \frac{\partial}{\partial y} V(y)\right]$$

Thus,

$$\frac{d}{dy} V(y) = E_o \quad \text{(ordinary diff. equation.)}$$

Solving this equation:

$$dV(y) = E_o\, dy$$

Take the integral of both sides,

$$\int dV(y) = \int E_o\, dy$$

or $\quad V(y) = E_o y + C$

If $V(y=0) = 0V \Rightarrow C = 0$

and $V(y) = E_o y$

Then, $\bar{F} = -\bar{\nabla} W_E = -\left(\hat{a}_x \cancel{\frac{d}{dx} W_E} + \hat{a}_y \frac{d}{dy} W_E + \hat{a}_z \cancel{\frac{d}{dz} W_E}\right)$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad 0 \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad 0$

where $W_E = \frac{1}{2} QV = \frac{1}{2} Q (E_o y)$

$$\Rightarrow \bar{F} = -\hat{a}_y \frac{d}{dy}\left(\frac{1}{2} Q E_o y\right) = -\hat{a}_y \frac{1}{2} Q E_o$$

or since $E_o = \frac{Q}{\epsilon S}$ (pg. 26)

$$\Rightarrow \bar{F} = -\hat{a}_y \frac{1}{2} \frac{Q^2}{\epsilon S} \quad (N)$$

In order to lift a 100 kg person on $1 m^2$ conductor, we need $F = 100(9.8) \cong 100(10) = 1000$ N.

If $S = 1 m^2$, $\epsilon = \epsilon_o$ (air filled), $d = 1 cm = 0.1 m$.

$$C = \frac{\epsilon_o S}{d} = \frac{(8.854 \times 10^{-12})(1)}{0.1} = 8.854 \times 10^{-11} \text{ F}$$

$$\Rightarrow C = \frac{Q}{V} \Rightarrow Q = CV = (8.854 \times 10^{-11}) V$$

Put everything into force formula

$$\bar{F} = -\hat{a}_y \frac{1}{2} \frac{Q^2}{\epsilon S} \quad (N)$$

or

$$1000 = \frac{1}{2} \frac{(8.854 \times 10^{-11})^2 \cdot V^2}{8.854 \times 10^{-12} \cdot (1)}$$

$\Rightarrow V = 1.5$ MV. is necessary.

Thus, keeping 100 kg person 1 cm above the ground, we need 1.5 million volts. This is not practical.

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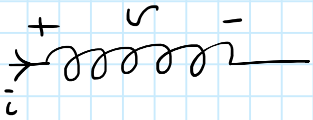
Magnetic Energy:

$W = \int \underbrace{v \cdot i}_{\text{Power}} \cdot dt$, v = voltage across an inductor

i = current through an inductor.



We know that

$$v = L \frac{di}{dt},$$

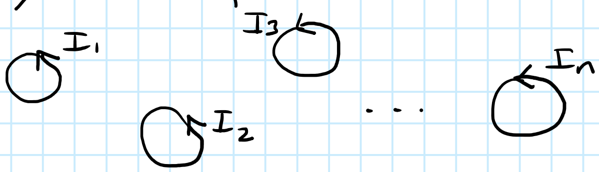
Then,

$$W = L \int_0^I i \, di = \boxed{\frac{1}{2} L I^2 \quad (J)}$$ (Total energy stored inside an inductor.)

Since $L = \frac{\phi}{I} \Rightarrow I = \frac{\phi}{L}$

Then, $$\boxed{W = \frac{1}{2} I \phi \quad (J)}$$

In a system of discrete inductances:



Total energy stored in this system:

$$\boxed{W_m = \frac{1}{2} \sum_{k=1}^{N} I_k \phi_k \quad (J)}$$ (Total energy stored in an static magnetic system of N currents)

In case of continuous distribution of current within a volume V,

$$\boxed{W_m = \frac{1}{2} \int_{V'} \mu H^2 \, dv \quad (J)}, \quad H = \frac{B}{\mu}$$

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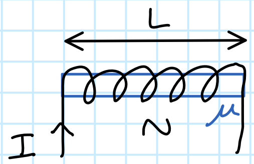
Ex:

Find the static magnetic energy stored in a solenoid inductor

Ans:



$n = \frac{N}{L}$ (# of turns in a unit length)

$$W_B = \frac{1}{2}\int_{V'} \mu H^2 dv$$

$$H = \frac{B}{\mu} = \frac{\cancel{\mu} n I}{\cancel{\mu}} = nI$$

$$\Rightarrow W_B = \frac{1}{2}\int_{V'} \mu n^2 I^2 dv = \frac{1}{2}\mu n^2 I^2 \underbrace{\int_{V'} dv}_{S.L}$$

$$\Rightarrow W_B = \frac{1}{2}\mu n^2 I^2 S L$$

$$= \frac{1}{2}\mu \frac{N^2}{L^{\cancel{2}}} \cdot S \cancel{L} I^2 = \frac{1}{2}\underbrace{\mu S \frac{N^2}{L}}_{L} I^2 = \frac{1}{2} L I^2$$

- Forces on Current Carrying Conductors -

Let us consider an element of conductor $d\ell$ with a cross sectional area S. If there are N electrons per unit volume moving with a velocity $\vec{u}$ in the direction of $d\ell$, then

$$dF_m = q\,\vec{u} \times \vec{B} = -Ne \underbrace{S\,d\ell}_{\text{Volume}}\, \vec{u} \times \vec{B}$$

where e = electron charge.

or

$$d\bar{F}_m = \underbrace{-NeS|\vec{u}|}\, \bar{d\ell} \times \bar{B}$$

$$I = \frac{NSe.\,x}{t} = \frac{q}{t} \text{ (A)}.$$

distance.

Then,

$$\boxed{d\bar{F}_m = \bar{I}\, d\ell \times \bar{B} \text{ (N)}}$$

Therefore,

$$\boxed{\bar{F}_m = I \oint_c \bar{d\ell} \times \bar{B} \text{ (N)}}$$

Consider two circuits carrying currents I_1 and I_2 respectively,

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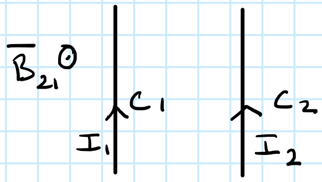


$\bar{B}_{21}$ is the mag. field caused by I_2.

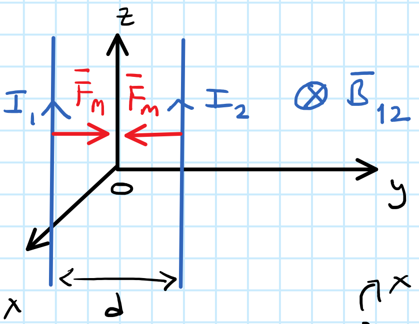
The force $\bar{F}_{21}$ on circuit C_1 is

$$\bar{F}_{21} = I_1 \oint_{C_1} \bar{dl}_1 \times \bar{B}_{21}$$

Ex:

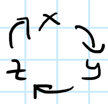
Determine the force per unit length between two infinitely long parallel conducting wires carrying currents I_1 and I_2 in the same direction. The wires are separated by a distance d.

Ans:



$$d\bar{F}_m = I_2(\hat{a}_z dz \times \bar{B}_{12})$$

where $\bar{B}_{12} = -\hat{a}_x \dfrac{\mu_o I_1}{2\pi d} \left(\dfrac{Wb}{m^2}\right)$



Substitute $\bar{B}_{12}$ into $d\bar{F}_m$ formula:

$$d\bar{F}_m = I_2 \hat{a}_z dz \times \left(-\hat{a}_x \frac{\mu_o I_1}{2\pi d}\right) = -\hat{a}_y \frac{\mu_o I_1 I_2}{2\pi d} dz \ (N)$$

- This is the differential force acting on differential length of the conductor dl. ($dl = dz$)

To find $\bar{F}_m$:

$$\bar{F}_m = \int d\bar{F}_m = \int_{z=0}^{z=L} -\hat{a}_y \frac{\mu_o I_1 I_2}{2\pi d} dz \ \left(\frac{N}{m}\right)$$

- This is the force per unit length of the conductor.

or

$$\boxed{\bar{F}_m = -\hat{a}_y \frac{\mu_o I_1 I_2 L}{2\pi d} \ (N)}$$

- The wires are attracted to one another.

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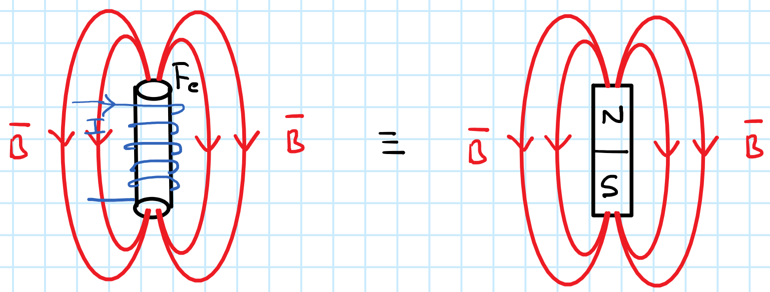
– The force between two wires carrying currents in opposite directions is repulsive.

Electromagnets:

Magnet is a material whose charges (electrons) spin even in the absence of a magnetic field.
Thus, magnet creates its own magnetic field.



Electromagnet. Magnet

If we can make a circuit that acts as a magnet. This circuit is called "electromagnet".

Forces in terms of Stored Magnetic Energy:

As we have studied in the electric field,

$$\bar{F} = -\bar{\nabla} W$$

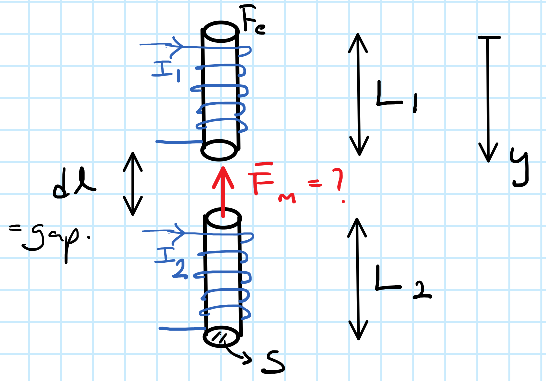
Then,

$$\boxed{\bar{F}_m = -\bar{\nabla} W_m} \quad (N)$$

Ex:

Consider a pair of electromagnets

Find $\bar{F}_m = ?$

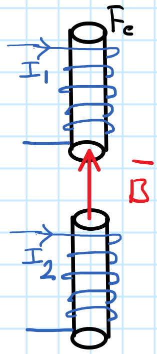


Ans:

$$\bar{F}_m = -\bar{\nabla} W_m$$

where

$$W_m = \frac{1}{2} \int_{v'} \frac{|B|^2}{\mu} dv \quad (J).$$



where

$$\bar{B}_1 = \frac{\mu N_1 I_1}{L_1} \left(\frac{Wb}{m^2}\right), \quad \bar{B}_2 = \frac{\mu N_2 I_2}{L_2} \left(\frac{Wb}{m^2}\right)$$

$$\bar{B}_{total} = \bar{B}_1 + \bar{B}_2 = 2\mu \frac{NI}{L} \quad (\text{for } N_1 = N_2,\ L_1 = L_2,\ I_1 = I_2)$$

$$\Rightarrow dW_m = \frac{1}{2} \frac{B^2}{\mu} \underbrace{S\,dy}_{dv} = \frac{1}{\cancel{2}} \cancel{4}^{2} \mu^{\cancel{2}} \frac{N^2 I^2}{L^2} \cdot \frac{1}{\cancel{\mu}} S\,dy = 2\mu n^2 I^2 S\,dy, \quad n = \frac{N}{L}.$$

$$\Rightarrow \bar{F}_m = -\bar{\nabla} W_m = -\hat{a}_y \frac{dW_m}{dy} = -\hat{a}_y \frac{2\mu n^2 I^2 S \cancel{dy}}{\cancel{dy}}$$

$$= -\hat{a}_y 2\mu n^2 I^2 S \quad (N)$$

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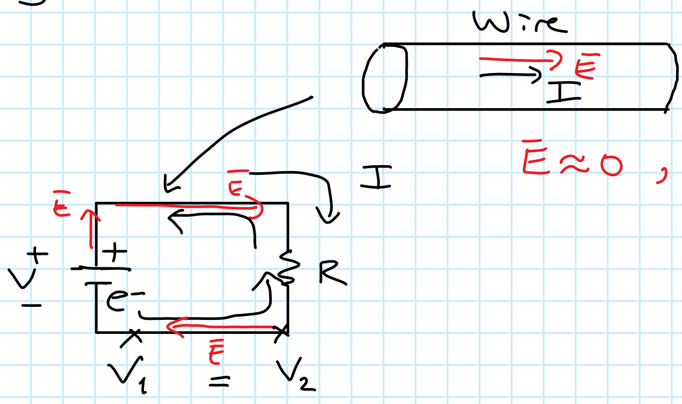
– Electric Circuits –

An Electric circuit is a connection of a source with resistors by conducting wires.



$E \approx 0$, $E \neq 0$

$+ \quad V \quad -$

$= V_1 - V_2 = 0$

$\boxed{\bar{J} = \sigma \bar{E}}$ (Ohm's law in point form)

where σ = conductivity

Also, $\boxed{\rho = \frac{1}{\sigma} = \text{Resistivity}}$

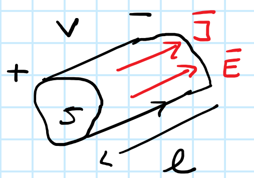
σ for some conductors:

$\sigma_{copper} = 5.8 \times 10^{7} \left(\frac{S}{m}\right)$, $\sigma_{Silicon} = 1.6 \times 10^{-3} \left(\frac{S}{m}\right)$, $\sigma_{rubber} = 10^{-5} \left(\frac{S}{m}\right)$

↳ metal ↳ semi-conductor ↳ dielectric.

Ex:

Find the resistance of a conductor given below.



Uniform $\bar{E}$ field is assumed.

Ans:

1-) Find E. $V = E \cdot l \Rightarrow E = \frac{V}{l}$

2-) Use $\bar{J} = \sigma \bar{E} \Rightarrow J = \sigma \left(\frac{V}{l}\right)$

Also, $J = \frac{I}{S}$

Then $\frac{I}{S} = \sigma \left(\frac{V}{l}\right)$ or $V = \underbrace{\frac{l}{\sigma S}}_{R = \text{resistance}} \cdot I$ (Ohm's law in volumetric form.)

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Power Dissipation:

$$p = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \underbrace{q\bar{E}}_{\bar{F}} \cdot \underbrace{\vec{v}}_{\frac{\Delta x}{\Delta t}}$$

(This is the power corresponding to moving a charge against the electric field.)

p → instanteneous power.

The power delivered to all charges in a volume dv is:

$$dP = \Sigma p = \bar{E}(\Sigma q\vec{v})\,dv$$

dv → volume element.

$$\Rightarrow dP = \bar{E} \cdot \bar{J}\,dv$$

Integrate both sides in volume V

$$P = \int_V \bar{E} \cdot \bar{J} \cdot dv \ \ (W) = \underbrace{\int_C \bar{E} \cdot d\bar{l}}_{= V} \cdot \underbrace{\int_S \bar{J} \cdot d\bar{s}}_{I} \ \ (W)$$

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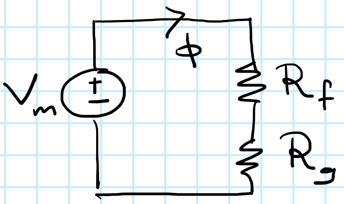
- Magnetic Circuits -

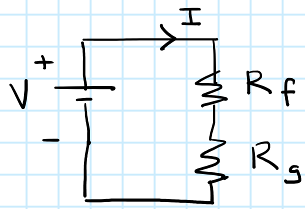
$$\oint_c \bar{H}\cdot d\bar{l} = NI = V_m = \text{Magnetic source.}$$

Magnetic Circuits	Electric Circuits
mmf, $V_m = NI$ (A)	emf, V_{emf}
Magnetic flux, Φ (Wb)	electric current, I
reluctance, $\mathcal{R}$ (H^{-1})	resistance, R
permeability, μ.	conductivity, σ

emf = Electro motive force.



Magnetic Circuit

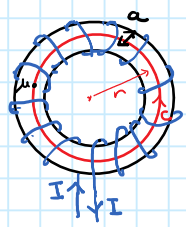


Electric circuit

$$\boxed{\text{Reluctance} = \frac{l_g}{\mu_0 S}}$$

, l_g = length of the inductor
S = Area of which the flux passes through.

For example, for a toroidal inductor



$$l_g = 2\pi r, \quad S = \pi a^2,$$

a = radius of the core.

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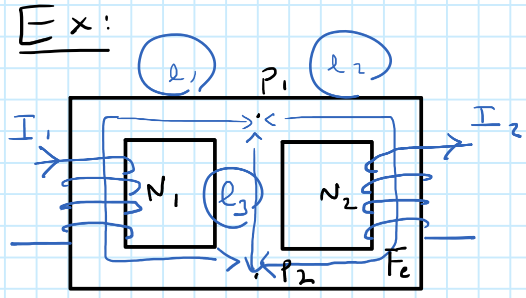
When solving magnetic circuits, we use the following formulas:

1-) $\sum_j N_j I_j = \sum_k R_k \phi_k$ (Corresponds to KVL)

2-) $\sum_j \phi_j = 0$ (corresponds to KCL)

Ex:



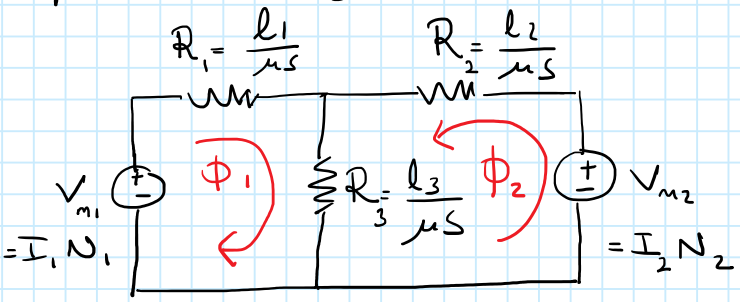
Suppose $\mu S = 2$, $l_1 = 2$, $l_2 = 2$, $l_3 = 1m$.

$N_1 = N_2 = 100$ turns.

$I_1 = I_2 = 10mA$. $\Rightarrow$ Find $\phi_1 = ?$

Ans:

The equivalent magnetic circuit is:



where

$R_1 = \frac{2}{2} = 1$

$R_2 = \frac{2}{2} = 1$

$R_3 = \frac{1}{2} = 0.5 \ (H^{-1})$

KVL for Loop 1:

$$N_1 I_1 = R_1 \phi_1 + R_3 (\phi_1 + \phi_2) \quad \text{—(1)}$$

KVL for loop 2:

$$N_2 I_2 = R_2 \phi_2 + R_3 (\phi_2 + \phi_1) \quad \text{—(2)}$$

(1) can be re-written as:

$$1 = \phi_1 + \frac{1}{2}\phi_1 + \frac{1}{2}\phi_2 \quad \text{—(3)}$$

(2) can be re-written as:

$$1 = \phi_2 + \frac{1}{2}\phi_1 + \frac{1}{2}\phi_2 \quad \text{—(4)}$$

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From (3) and (4):

$$\frac{3}{2}\phi_1 + \frac{1}{2}\phi_2 = 1 \Rightarrow \overset{-3/}{} \ 3\phi_1 + \phi_2 = 2$$

$$-9\phi_1 - 3\phi_2 = -6$$

$$\frac{3}{2}\phi_2 + \frac{1}{2}\phi_1 = 1 \Rightarrow 3\phi_2 + \phi_1 = 2$$

$$-8\phi_1 + \phi_1 = -4$$

$$\Rightarrow \phi_1 = \frac{-4}{-8} = \frac{1}{2} \text{ Wb.}$$